

A Simple Proof of the Quadratic Formula

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Abstract

This article provides a simple proof of the quadratic formula, which also produces an efficient and natural method for solving general quadratic equations. The derivation is computationally light and conceptually natural, and has the potential to demystify quadratic equations for students worldwide.

1 Introduction

The quadratic formula was a remarkable triumph of early mathematicians, marking the completion of a long quest to solve quadratic equations, with a storied history stretching as far back as the Old Babylonian Period around 2000–1600 B.C. [18, 21]. Over four millennia, many recognized names in mathematics left their mark on this topic, and the formula became a standard part of a first course in Algebra.

However, it is unfortunate that for billions of people worldwide, the quadratic formula is also their first experience of a rather complicated formula which they memorize. Many typically learn it as the systematic alternative to a guess-and-check method that only factorizes certain contrived quadratic polynomials. Countless mnemonic techniques abound, from stories of negative bees considering whether or not to go to a radical party, to songs set to the tune of *Pop Goes the Weasel*. A derivation by completing the square is usually included in the curriculum, but its motivation is often challenging for first-time Algebra learners to follow, and its written execution can be cumbersome.

This article introduces an independently discovered simple derivation of the quadratic formula, which also produces a computationally-efficient and natural method for solving general quadratic equations. The author would actually be very surprised if this pedagogical approach has eluded human discovery until the present day, given the 4,000 years of history on this topic, and the billions of people who have encountered the formula and its proof. Yet this technique is certainly not widely taught or known. After an earlier version of this arXiv preprint was publicized by formal and informal media, the author was contacted by many people with potential references. However, the author still has not found a previously-existing publicly-shared reference detailing this pedagogical approach which is mathematically complete and formally correct. (Similar writings which come close, most notably Savage [22], are outlined in Section 3.3.) This article aims to provide a safely referenceable method and derivation which is logically sound. That said, it is entirely possible that

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there is still a previously-existing reference waiting to be found. So, this article seeks at the very least to popularize a delightful alternative pedagogical approach to solving quadratic equations, which is practical for integration into all mainstream curricula.

2 Derivation

The reader is encouraged to remember what it was like to be a first-time Algebra learner, where it took some concentration to combine fractional expressions, especially those including multiple variables and constants. This section is intentionally written at that level of simplicity, to emphasize how straightforward all the algebraic manipulations and concepts are.

2.1 Computationally simple derivation of an explicit root formula

Throughout this article, we work over the complex numbers. The starting point is that if we can find a factorization of the following form

$$x^2 + Bx + C = (x - R)(x - S), \tag{1}$$

then a value of x makes the product equal zero precisely when at least one of the factors becomes zero, which happens precisely when $x = R$ or $x = S$. By the distributive law, it suffices to find two numbers R and S with sum $-B$ and product C ; then, $\{R, S\}$ will be the complete set of roots.¹

Two numbers sum to $-B$ precisely when their average is $-\frac{B}{2}$, and so it suffices to find two numbers of the form $-\frac{B}{2} \pm z$ which multiply to C , where z is a single unknown quantity, because they will automatically have the desired average.² (If z turns out to be 0, then we factor with $R = S = -\frac{B}{2}$.) The product $(-\frac{B}{2} + z)(-\frac{B}{2} - z)$ conveniently matches the form of a difference of squares, and equals C precisely when

$$\left(-\frac{B}{2}\right)^2 - z^2 = C,$$

or equivalently, precisely when we have a z which satisfies

$$z^2 = \frac{B^2}{4} - C.$$

Since a square root always exists (extending to complex numbers if necessary), arbitrarily select a choice of square root of $\frac{B^2}{4} - C$ to serve as z , in order to satisfy the last equation. Tracing back through the logic, we conclude that the desired R and S exist in the form $-\frac{B}{2} \pm z$, and so

$$-\frac{B}{2} \pm \sqrt{\frac{B^2}{4} - C} \tag{2}$$

are all the roots of the original quadratic. □

¹This is the standard method of factoring, which corresponds to the converse of relations that are typically attributed to Viète [25]. Those relations provide inspiration for this method, but are not required for logical completeness, because the converse is a straightforward consequence of the distributive property. By using the converse, this proof does not rely on the theorem that two roots (counting multiplicity) always exist.

²This substitution, and this entire solution to find two numbers given their sum and product, was known to the Babylonians (see, e.g., Burton [4], Gandz [12], Irving [17], or Katz [18]). It also appeared in the first book of Diophantus [9]. This approach therefore represents the fusion of these ancient techniques together with Renaissance-era mathematical sophistication. Further historical context follows later in this article.

2.2 Example of use as a method

The computational and conceptual simplicity of this derivation actually renders it unnecessary to memorize any formula at all, even for general coefficients of x^2 . The proof naturally transforms into a method, and students can execute its logical steps instead of plugging numbers into a formula that they do not fully understand. Consider, for example, the following quadratic:

$$\frac{x^2}{2} - x + 2 = 0.$$

Multiplying both sides by 2 to make the coefficient of x^2 equal to 1, we obtain the equivalent equation

$$x^2 - 2x + 4 = 0.$$

If we find two numbers with sum 2 and product 4, then they are all the solutions. Two numbers have sum 2 precisely when they have average 1. So, it suffices to find some z such that two numbers of the form $1 \pm z$ have product 4 (their average is automatically 1). The final condition is equivalent to each of these equivalent equations:

$$\begin{aligned}1 - z^2 &= 4 \\ z^2 &= -3.\end{aligned}$$

We can satisfy the last equation by choosing $i\sqrt{3}$ for z . Tracing back through the logic, we conclude that $1 \pm i\sqrt{3}$ are all the solutions to the original quadratic. Irrational and imaginary numbers pose no obstacle to this method.

2.3 Derivation of traditional quadratic formula with arbitrary x^2 coefficient

If one specifically wishes to derive the commonly memorized quadratic formula using this method, one only needs to divide the equation $ax^2 + bx + c = 0$ by a (assume nonzero) to obtain an equivalent equation which matches the form of (1):

$$x^2 + \left(\frac{b}{a}\right)x + \left(\frac{c}{a}\right) = 0.$$

Plugging $\frac{b}{a}$ and $\frac{c}{a}$ for B and C in (2), the roots are:

$$-\frac{b}{2a} \pm \sqrt{\frac{b^2}{4a^2} - \frac{c}{a}} = -\frac{b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{4a^2}} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

□

Observe that with this approach, all of the useful and interesting conceptual insights are fully isolated in a computationally light derivation of an explicit formula, while also producing an efficient and understandable algorithm. The routine but laborious computational portion is required only if a general formula is sought for memorization purposes. In light of the efficient algorithm, however, it becomes questionable whether there is merit to memorizing a formula without understanding. For example, although the solution to a general linear equation $ax + b = 0$ is $x = -\frac{b}{a}$ (assume $a \neq 0$), the equation is typically solved via manipulation instead of plugging into a memorized formula.

3 Discussion

3.1 Practical relation to other curricular concepts

Before learning the quadratic formula, students learn how to multiply binomials, and they see useful expansions such as $(u + v)^2 = u^2 + 2uv + v^2$ and $(u + v)(u - v) = u^2 - v^2$. Indeed, the first of these expansions is the cornerstone of the traditional proof of the quadratic formula by completing the square. The second of these expansions is also of wide importance: among other things, it is eventually used to rationalize the denominator of expressions such as $\frac{1}{\sqrt{3}-\sqrt{2}}$ by multiplying the numerator and denominator by $\sqrt{3} + \sqrt{2}$.

Our approach shows that the factoring method can always be made to work. It always produces two roots (counting multiplicity) whose sum and product correspond to coefficients of the quadratic. This therefore presents an opportunity to prove Viète's sum and product relations for quadratics.

For first-time Algebra learners, the only new leap of insight is that if one is seeking two numbers with a desired sum, then they can be parameterized by their desired average, plus or minus a common unknown amount. In the modern day, a similar parameterization appears as a useful trick for mentally calculating products via the difference of squares, such as

$$43 \times 37 = (40 + 3)(40 - 3) = 40^2 - 3^2 = 1591.$$

This is an ancient trick. Some historians believe the Babylonians used it thousands of years ago, multiplying in their base-60 number system by subtracting from tables of squares (see, e.g., Derbyshire [7]). It was then natural for them to develop the same parameterization for finding two numbers, given their sum and product.

3.2 Comparison to completing the square

The most common proof of the quadratic formula is via completing the square, and that was also the method used by al-Khwarizmi [1] in his systematic solutions to abstract quadratic equations. Compared to our approach, the motivation is less direct, as the step of completing the square (for the simple situation of $x^2 + Bx + C = 0$) simultaneously combines three insights:

- (i) The x^2 and Bx can be entirely absorbed into a square of the form $(x + D)^2$ by using only part of the expansion $(u + v)^2 = u^2 + 2uv + v^2$ “backwards,” to attempt to factor an expression that begins with $u^2 + 2uv$.
- (ii) This perfect square can be created by adding and subtracting the appropriate constant, which is $(\frac{B}{2})^2$.
- (iii) After these manipulations are complete, the equation will have $(x + \frac{B}{2})^2$ and some constants, and any such equation can be solved by moving constants around and taking a square root.³

The full combination of these insights is required to understand the motivation for why one should even write down the specific offsetting quantities $+\frac{B^2}{4} - \frac{B^2}{4}$ in the first line of the completing the square:

$$x^2 + Bx + \frac{B^2}{4} - \frac{B^2}{4} + C = 0.$$

³ It is interesting to note that this step of completing the square uses the fact that the complete set of solutions to $x^2 = K$ is $\{\pm\sqrt{K}\}$, which is not obvious to a first-time student of quadratics. In contrast, our approach only requires that there exists an explicit choice of x which satisfies $x^2 = K$, which is often explicitly constructable.

In contrast, our approach starts from students' existing experience searching for a pair of numbers with given sum and product, which naturally arises during the factoring method. It shows them that the (sometimes frustrating) guess-and-check process can be replaced by one idea: to parameterize the pair by its average plus or minus a common unknown offset. No particular formula needs to be written for the offset itself (unlike the case of carefully selecting $\frac{B^2}{4}$), and we can simply call it an unknown z . Instantly, their previous experience of trial and error is replaced by a "forward" expansion of the form $(u + v)(u - v) = u^2 - v^2$, which produces an exciting lone z^2 , revealing the pair of numbers with all guesswork eliminated.

3.3 Brief historical context

Could such a simple proof and pedagogical method possibly be new? The author researched the English-language literature on the history of mathematics, and consulted English translations of old manuscripts, from mathematical traditions ranging from Diophantus [9] to Brahmagupta [3], Yanghui [26], and al-Khwarizmi [1]. This section is too brief to do full justice to the history, and mainly serves to point the interested reader to relevant resources with much richer detail. In particular, several books have surveyed the topic of the quadratic formula, such as Chapter 2 of Irving [17], and mathematical history books such as Burton [4], Derbyshire [7], and Katz [18].

As preserved in their cuneiform tablets, the Babylonians had evidence of formulas for a wide variety of problems of quadratic nature, dating back to the Old Babylonian Period around 2000–1600 B.C. Although today we can easily use substitution to reduce them to standard one-variable quadratic equations, the Babylonians did not have a way to solve those standard quadratics. However, they did consider the problem of finding the dimensions of a rectangular field given its semiperimeter and area, and had the key substitution used in our solution method. This is discussed in Gandz's extensive 150-page study of quadratic equations [12], as well as in Berriman [2], Burton [4], Gandz [13], Katz [18], and Robson [21]. The ancient Egyptians also had evidence of work with a two-term quadratic equation, preserved on scraps of a Middle Kingdom papyrus [10].

Ancient Chinese mathematicians had solutions to practical problems of quadratic nature, such as Problem 20 in Chapter 9 of *Jiu Zhang Suan Shu* (The Nine Chapters on the Mathematical Art), which was written over several centuries and completed around 100 A.D. Practical problems of quadratic nature continued to be considered by other Chinese mathematicians, such as the 13th-century Yang Hui. See, e.g., the book in Chinese by Zeng [27] or the book in English by Lam [19].

The Greeks had several methods of approaching certain types of quadratic problems, both algebraic and geometric, as surveyed in Eells [11]. Heath's translation [15] of Diophantus [9] from around 250 A.D. clearly shows the solution of the core problem of finding two numbers with given sum and product (Book I Problem 27), using the key parameterization in terms of the average.

Indian mathematicians also had derived a formula for quadratics. Although Brahmagupta [3] did not discover it himself, one root of the quadratic formula (without derivation) appears in his writings circa 628 A.D. See, e.g., the translation by Colebrooke [6] or the commentary by Sharma et al. [23]. A derivation due to Sridhara from around 900 A.D. appears in Puttaswamy [20].

The Persian mathematician al-Khwarizmi published his influential work [1] around 825 A.D., where he abstractly considered and solved the general form of quadratic equations, without starting from practical applications. His work split into several cases, because he did not allow numbers to be negative or zero. Consequently, his formulas did not produce all roots, although they did produce all roots according to the standards of what a number was at the time.

As mathematics in Western Europe flourished during the Renaissance, successive formulations and proofs appeared, from Stevin [24] to Viète [25] and Descartes [8], ultimately taking on the modern form that we know today. In the years since then, new proofs have occasionally appeared, such as two in *The American Mathematical Monthly*: Heaton [16] in 1896 and Cirul [5] in 1937.

After an earlier version of this arXiv preprint circulated across the Internet, references of more recent similar work were identified. The most similar is Savage [22]. His approach essentially overlapped in almost all calculations, but had a pedagogical difference in choice of sign, factoring in the form $(x + p)(x + q)$ and negating at the end. Perhaps due to a friendly writing style, that published article has some reversed directions of implication that are not formally correct. The directional reversals brought in the same extra assumption as in Footnote 3 when completing the square, creating another pedagogical difference. That said, those oversights can easily be corrected by using language similar to our presentation. Gowers [14] also had happened upon a similar approach, while informally presenting a natural way to deduce the cubic formula. As he was writing for a different purpose, his version as written uses Viète's sum and product relations at the outset, requiring initial knowledge that there always exist two roots (a pedagogical difference for first-time Algebra learners), and deducing forward. It can easily be converted to avoid this existence assumption by using factoring as in our approach.

In summary, the author has not yet found a previously-existing book or paper which states the same pedagogical method as this present article and precisely justifies the steps, but there exist independent references that contain the key ideas and can be adapted to achieve this. That said, it is entirely possible that the method in this present article was previously observed by people who did not share their findings.

3.4 Why not centuries ago?

The two main components of our derivation have existed for hundreds of years (polynomial factoring converse of Viète's relations) and for thousands of years (Babylonian solution to the sum-product problem). Furthermore, the reduction from the Babylonian problem to a standard quadratic equation has been well-known for an extremely long time. Even al-Khwarizmi [1], after abstractly analyzing general quadratic equations, showed how to use his formula to find two numbers with sum 10 and product 21. Like many students in the modern era, he used substitution to reduce the problem to a single-variable quadratic equation, and solved it with the quadratic formula. Why, then, didn't early mathematicians just reverse their steps and find our simple derivation?

Perhaps the reason is because it is actually mathematically nontrivial to attempt to factor $x^2 + Bx + C = (x - R)(x - S)$ over complex numbers. Even if the original quadratic polynomial has real coefficients, it is sometimes impossible to find two real numbers with sum $-B$ and product C . Early mathematicians did not know how to reason with a full (algebraically closed) system of numbers. Indeed, al-Khwarizmi did not even use negative numbers, nor did Viète, not to mention the complex numbers that might arise in general. Perhaps, by the time our mathematical sophistication had advanced to a sufficient stage, the Babylonian trick had faded out of recent memory, and we already found the method of completing the square to be sufficiently elementary for integration into mainstream curriculum.

It is worth noting that the author discovered the solution method in this paper in the course of filming mathematical explanations, to explain advanced concepts to particularly young students. Given his audience, he was systematically going through the middle school math curriculum, creating alternative explanations in elementary language. To prepare students for the mindset of

factoring, he posed a standalone sum-product problem, designed to be solved via guess-and-check. While teaching it one evening, his background in coaching math competition students led him to independently reinvent the Babylonian parameterization in terms of the average, and to recognize the difference of squares. Later, when teaching factoring, he suddenly realized that the same technique worked in general, leading to a simple proof of the quadratic formula! May this story encourage the reader to think afresh about old things; seeing as how progress was made on this 4,000 year old topic, more surprises certainly await the light of discovery.

References

- [1] Al-Khwarizmi, Muhammad ibn Musa. (circa 825). *Al-kitab al-mukhtasar fi hisab al-gabr wa'l-muqabala*.
- [2] Berriman, A. E. (1956). The Babylonian Quadratic Equation. *The Mathematical Gazette*. 40(333): 185–192. doi.org/10.2307/3608807
- [3] Brahmagupta. (circa 628). *Brahma Sphuta Siddhanta*.
- [4] Burton, David M. (2010). *History of Mathematics, An Introduction*, 7th ed. New York, NY: McGraw-Hill.
- [5] Cirul, J. W. (1937). A Method for Solving Quadratic Equations. *The American Mathematical Monthly*. 44(7): 462–463. doi:10.2307/2301134
- [6] Colebrooke, Henry Thomas. (1817). *Algebra, with Arithmetic and Mensuration from the Sanscrit of Brahmagupta and Bhascara*. London, United Kingdom: John Murray, Albemarle Street.
- [7] Derbyshire, John. (2006). *Unknown Quantity: A Real and Imaginary History of Algebra*. Washington, DC: Joseph Henry Press.
- [8] Descartes, René. (1637). *La Géométrie*.
- [9] Diophantus. (circa 250). *Arithmetica*.
- [10] Edwards, I. E. S., Gadd, C. J., and Hammond N. G. L., Editors. (1971). *The Cambridge Ancient History: Early History of the Middle East, Volume 1 Part 2*. Cambridge, United Kingdom: Cambridge University Press.
- [11] Eells, Walter C. (1911). Greek Methods of Solving Quadratic Equations. *The American Mathematical Monthly*. 18(1): 3–14. doi:10.2307/2972623
- [12] Gandz, Solomon. (1937). The Origin and Development of the Quadratic Equations in Babylonian, Greek, and Early Arabic Algebra. *Osiris*. 3: 405–557.
- [13] Gandz, Solomon. (1940). Studies in Babylonian Mathematics III: Isoperimetric Problems and the Origin of the Quadratic Equations. *Isis*. 32(1): 103–115.
- [14] Gowers, William Timothy. <https://gowers.wordpress.com/2007/09/15/discovering-a-formula-for-the-cubic/>

- [15] Heath, Thomas L. (1910). *Diophantus of Alexandria: A Study in the History of Greek Algebra*, 2nd ed. Cambridge, United Kingdom: Cambridge University Press.
- [16] Heaton, Henry. (1896). A Method of Solving Quadratic Equations. *The American Mathematical Monthly*. 3(10): 236–237. doi:10.2307/2971099
- [17] Irving, Ronald S. (2013). *Beyond the Quadratic Formula*. Washington, DC: Mathematical Association of America.
- [18] Katz, Victor J. (2008). *A History of Mathematics*, 3rd ed. Boston, MA: Pearson.
- [19] Lam, Lay Yong. (1977). *A Critical Study of the Yang Hui Suan Fa: A Thirteenth-century Chinese Mathematical Treatise*. Singapore: Singapore University Press.
- [20] Puttaswamy, T. K. (2012). *The Mathematical Achievements of Pre-Modern Indian Mathematicians*. London, United Kingdom: Elsevier.
- [21] Robson, Eleanor. (2008). *Mathematics in Ancient Iraq: A Social History*. Princeton, NJ: Princeton University Press.
- [22] Savage, John. (1989). *Factoring Quadratics*. *The Mathematics Teacher*. 82(1): 35–36.
- [23] Shri Ram Swarup Sharma, Shri Mukund Mishra, Shri Vishwanath Jha, Shri Daya Shankar Dikshita, Shri Om Datt Sharma Shastri, Editors. (1966). *Shri Brahmagupta Viracita Brahma-Sphuta Siddhanta with Vasana, Vijnana and Hindi Commentaries Vol. I*, Delhi, India: Indian Institute of Astronomical and Sanskrit Research.
- [24] Stevin, Simon. (1585). *L'Arithmétique*.
- [25] Viète, François. (1579). *Opera Mathematica*. Edited by Frans van Schooten. Reprinted Leiden, Netherlands: Bonaventure and Abraham Elzevier, 1646.
- [26] Yang, Hui. (circa 1275). *Tian Mou Bi Lei Cheng Chu Jie Fa (Practical Rules of Arithmetic for Surveying)*.
- [27] Zeng, Hailong. (2006). *Jiu Zhang Suan Shu (The Nine Chapters on the Mathematical Art)*, modern Chinese translation and commentary. Chongqing, China: Chongqing University Press.