Part III

Hyperbolic Geometry

”... I bought a little blue cap, such as all the men and boys of the people wear down there, almost like a fez—the *boina*. I shall wear it in the rest-cure, and other places, perhaps. Monsieur shall judge if it becomes me.”

”What monsieur?”

”Sitting here in this chair.”

”Not Mynheer Peeperkorn?”

”He has already pronounced judgment—he says I look charming in it.”

”He said that—all of it? Did he really finish the sentence, so it could be understood?”

”Ah! It seems Monsieur is out of temper? Monsieur would be spiteful, cutting? He would laugh at people who are much greater and better, and—more *hu*—man than himself and his—his *ami bavard de la Méditerranée, son maître et grand parleur*—put together. But I cannot listen—”

Thomas Mann *The Magic Mountain*

1 Hyperbolic Geometry in the Poincaré Model

1.1 Inversion

Definition 1.1 (The inverse point). Given is a circle ∂D of radius R. For any given point P, we define the *inverse point* P' with respect to this circle to be the point *on the ray* $\overrightarrow{OP}$ such that $|OP| \cdot |OP'| = R^2$. A point on ∂D is its own inverse. For the center O itself, the inverse point is the point at infinity.

Problem 1.1. *Do a example for the construction of the inverse point.*

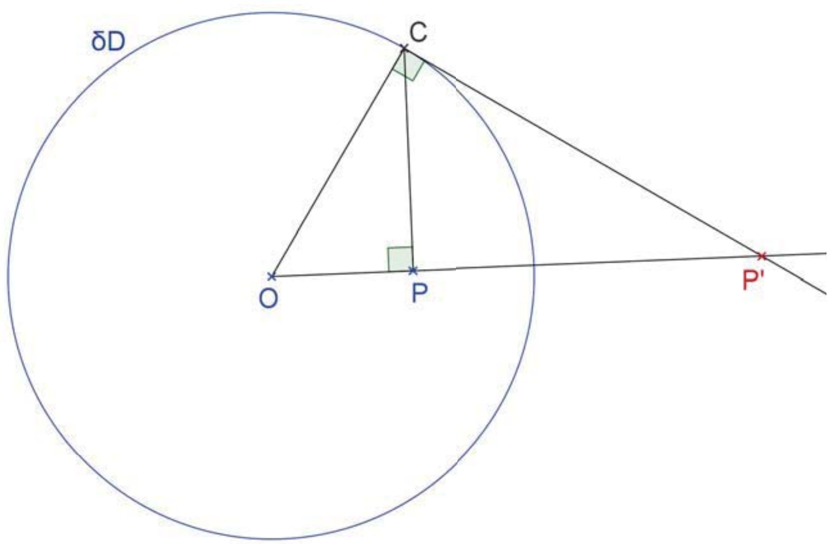

Figure 1.1: Construction of the inverse point

Construction 1.1 (The inverse point). *To construct the inverse of a given point P, one uses the the theorems related to the Pythagorean Theorem. One erects the perpendicular h to radius OC at point P. This step is different from the elliptic case! Let C an intersection of the perpendicular with ∂D. Next one erects the perpendicular on PC at point C, and gets a tangent to circle ∂D. The inverse point P' is the intersection at that tangent with the ray OP. Indeed, by the leg theorem, $|OP| \cdot |OP'| = |OC|^2 = R^2$.*

Remark. Alternatively, you can use the Thales' circle with diameter CP'. By converse Thales' theorem, P lies on that circle. Then use the chord theorem Euclid III.36 and get again, same as by leg theorem: $|OP| \cdot |OP'| = |OC|^2 = R^2$.

1.2 Points and lines in the Poincaré model

We explain now the Poincaré disk model of hyperbolic geometry. The reader should recall the basic idea of a model in mathematics, as explained in the passage *General* *remark about models in mathematics.* The Poincaré disk model of hyperbolic geometry is built in the Euclidean plane. The Euclidean geometry of the plane is the accepted ambient underlying reality, which one can call the "background ontology". Some cleverly chosen objects and relations from Euclidean geometry are now interpreted as objects and relations of hyperbolic geometry.

Hence we have in the Poincaré's disk model the ambient Euclidean plane as the accepted basic reality. In this plane one builds the hyperbolic geometry as a new secondary level. In order to stress this new situation and distinguish the two levels, I use quotation marks for the notions of hyperbolic geometry.

Definition 1.2 (Poincaré's disk model). We denote the open unit disk by

$$D = \{\, (x, y) : x^2 + y^2 < 1 \,\}$$

The center of D is denoted by O. Its boundary is

$$\partial D = \{\, (x, y) : x^2 + y^2 = 1 \,\}$$

The circle ∂D is also called the *line at infinity.*

- The points of D are the `"points"` for Poincaré's model.
- The points of ∂D are called `"ideal points"` or `"endpoints"`. The ideal points are not points for the hyperbolic geometry. Once the hyperbolic distance is introduced, they turn out to be infinitely far away.
- The `"lines"` for Poincaré's model are circular arcs perpendicular to ∂D, open at their ideal ends.

Problem 1.2. *Construct the hyperbolic line through two given points, in the Poincaré disk model.*

The construction uses the inverse point and the polar elements, as now explained.

Lemma 1.1. *If a circle passes through a pair of inverse points, it consists entirely of pairs of inverse points, and it intersects ∂D perpendicularly.*

A circle is a hyperbolic line if and only if it passes through a pair of inverse points P and P'.

Proof of the Lemma. Assume that a circle $\mathcal{C}$ passes through a pair of inverse points, P and P'. Let Q be a third point on the circle. We draw the rays $\overrightarrow{OPP'}$ and $\overrightarrow{OQ}$. Let Q_2 be the second intersection point of the ray $\overrightarrow{OQ}$ with circle $\mathcal{C}$. Because of the theorem of chords (Euclid III.35), we know that

$$|\,OP| \cdot |\,OP'| = |\,OQ| \cdot |\,OQ_2|$$

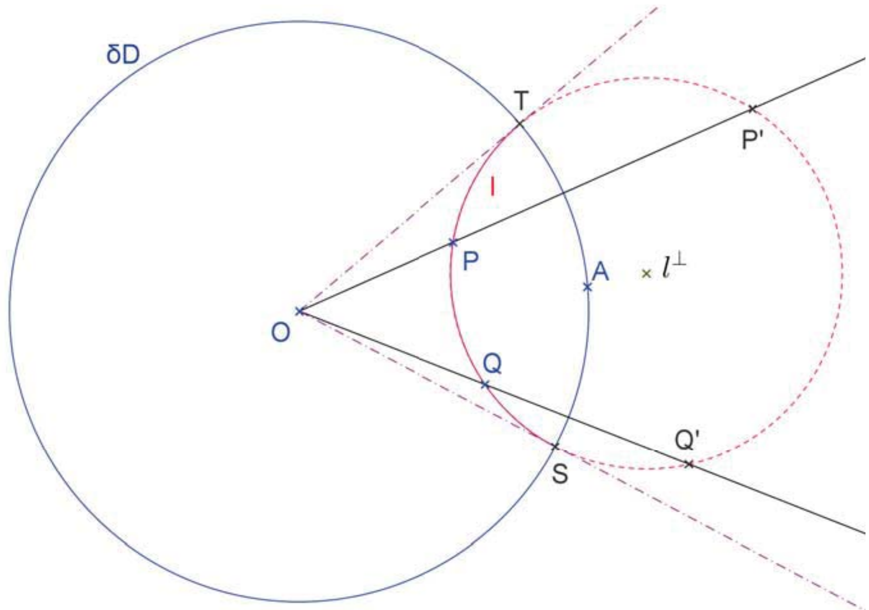

Figure 1.2: A orthogonal circle consists of pairs inverted points

By definition of inverse points

$$|OP| \cdot |OP'| = 1$$

and hence

$$|OQ| \cdot |OQ_2| = 1$$

The point O lies *outside* the circle $\mathcal{C}$, and hence Q and Q_2 lie on the same side of O. This confirms that the second intersection point $Q_2 = Q'$ is the inverse point of Q.

Why do the circles $\mathcal{C}$ and ∂D intersect perpendicularly? The center O lies outside of $\mathcal{C}$, because circle $\mathcal{C}$ contains a pair of inverse points, We construct a tangent from point O to circle $\mathcal{C}$ (Just one of the two tangents suffices.) Let T be the point at which the tangent touches circle $\mathcal{C}$. By construction $OT \perp Tl^\perp$. We apply the theorem of chords (Euclid III.36) for circle $\mathcal{C}$. Thus we get

$$|OP| \cdot |OP'| = |OT|^2$$

By definition of inverse points

$$|OP| \cdot |OP'| = 1$$

and hence

$$|OT|^2 = 1$$

This shows that the touching point T lies on ∂D, as well as on $\mathcal{C}$. Since $OT \perp Tl^\perp$ this implies that $Tl^\perp$ is a tangent to circle ∂D. By construction, OT is a tangent to circle l. Hence we have got two perpendicular tangents to the two circles l and ∂D at their intersection point T. This reasoning has shown that a circle passing through a pair of inverse points is a hyperbolic line.

The converse can be checked, too. We assume l is a hyperbolic line, and show that l contains a pair of inverse points. Let T be an intersection point of the two circles ∂D and l. Hence

$$|OT|^2 = 1$$

By definition of hyperbolic line, the two circles l and ∂D intersect perpendicularly, Hence the segment OT is a radius of circle ∂D as well as a tangent to circle l. The tangent from center O to the arc l touches circle l at point T. Let P be any point of l, and let P" be the second intersection point of ray $\overrightarrow{OP}$ with the circle l. By Euclid's theorem of chords

$$|OT|^2 = |OP| \cdot |OP"|$$

But by the definition of inverse points

$$|OP| \cdot |OP'| = 1$$

The three equations imply $|OP'| = |OP"|$. Hence $P' = P"$, and the inverse point P' lies on circle l, too. Hence circle l goes through the pair of inverse points P and P'. □

Definition 1.3 (The polar of point and line). The perpendicular bisector of P and P' is called the *polar* of point P. It is denoted by $P^\perp$ or P_{perp}. The polar $l^\perp$ of a hyperbolic line l is the (Euclidean) center of the circular arc l. The polar elements—$P^\perp$ for a point P, and $l^\perp$ for a line l—always lie *outside* the disk D. [Sometimes, we denote the midpoint of the segment PP' by K'. See Klein's model for the reason.]

The definitions of the polar for points and lines are consistent with incidence:

Fact. A point P lies on a hyperbolic line l if and only if the polar $P^\perp$ goes through the polar $l^\perp$.

Reason. Assume point P lies on the hyperbolic line l. Since l is a hyperbolic line, the arcs l and ∂D intersect perpendicularly. By the Lemma, this implies the arc l goes through both point P and the inverse point P'. Hence the center $l^\perp$ of the arc l lies on the perpendicular bisector of P and P', which we defined to be the polar $P^\perp$.

Conversely, if the polar $P^\perp$ goes through $l^\perp$, the center $l^\perp$ of the arc l has equal distances from P and P'. Hence the pair of inverse points P and P' lie on l. By the Lemma, l is a hyperbolic line. □

Construction 1.2 (Construct of a hyperbolic line through two given points). *Given are two points P and Q. To get the hyperbolic line l through P and Q, one constructs the polar $P^\perp$ and $Q^\perp$. Their intersection point is the polar $l^\perp$ of the line through P and Q.*

Question. What happens in the special case that $P^\perp$ and $Q^\perp$ are parallel? How does one get the hyperbolic line PQ in that special case?

Answer. The radius OP is perpendicular to $P^{\perp}$, and, similarly, the radius OQ is perpendicular to $Q^{\perp}$. The lines $P^{\perp}$ and $Q^{\perp}$ are parallel if and only if the two radii OP and OQ are parallel. This happens if and only if the three points P, Q and the center of the disk O lie on a Euclidean line .

In the special case that $P^{\perp}$ and $Q^{\perp}$ are parallel, the Euclidean line through the three points P, Q and the center of the disk O is the hyperbolic line through P and Q, too. Indeed , this line is a diameter of the disk D and hence perpendicular to ∂D.

Problem 1.3. *Do the construction of a hyperbolic line through two given points, for an example.*

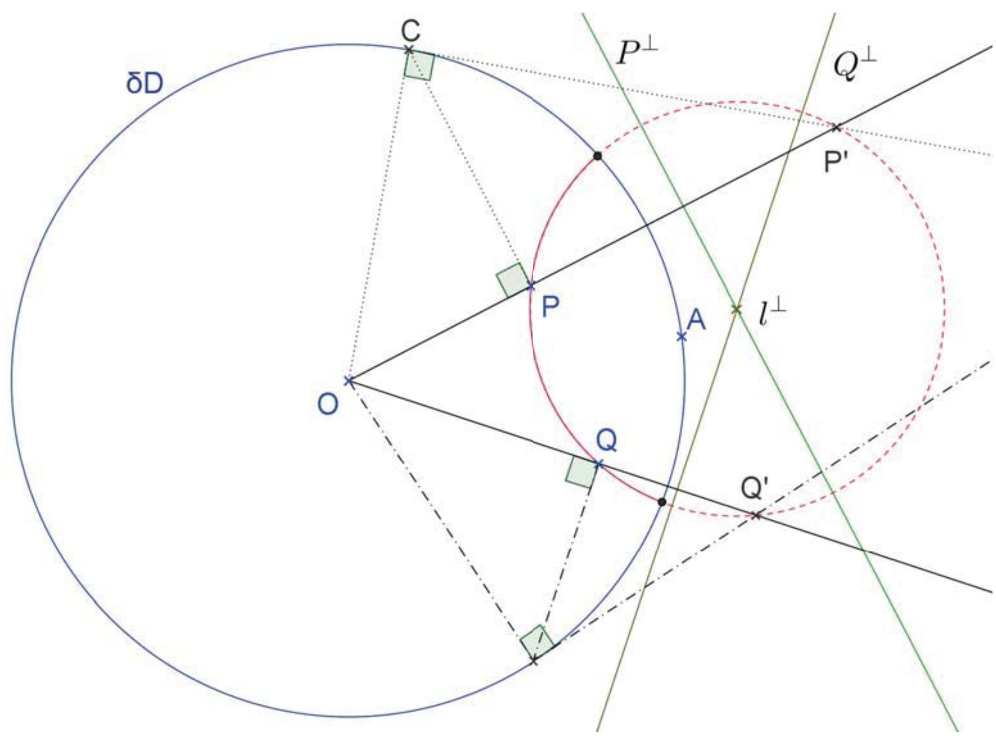

Figure 1.3: Construction of a hyperbolic line

Problem 1.4. *Complete the following sentences, referring to the drawing below.*

(a) The point D has the inverse image point A.

(b) The Euclidean line l is the polar of point A.

(c) The hyperbolic line α has the polar C.

(d) Point O is mapped to B via a reflection by the hyperbolic line α.

(e) Point B has as polar the Euclidean line $B^{\perp}$, that is not drawn.

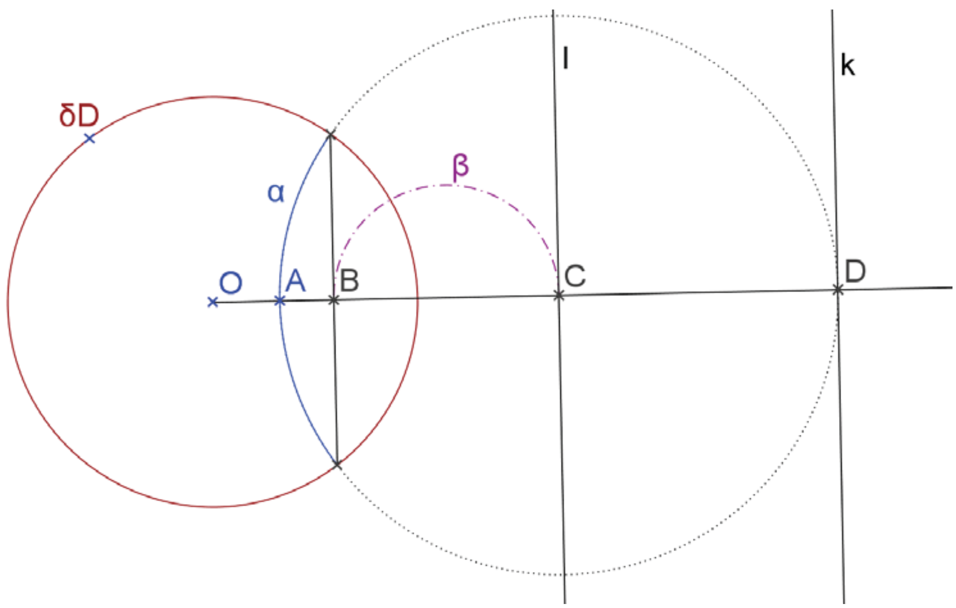

Figure 1.4: Recognize inverse and polar elements.

(f) Since B lies on β, the polar $B^{\perp}$ goes through the point $\beta^{\perp}$.

(g) Since A lies on α, the polar $A^{\perp}$ goes through the point $\alpha^{\perp}$.

(h) The polar of point A is line l.

(g) The polar of line α is point C.

(h) Indeed, line l goes through the point C.

1.3 Introduction of metric properties

Definition 1.4. The `"angles"` for Poincaré's model are the usual Euclidean angles between tangents to the circular arcs.

For the definition of a hyperbolic distance, one needs the cross ratio. The *cross ratio* of four point A, B, P, Q is defined as

$$(AB, PQ) = \frac{|AP| \cdot |BQ|}{|BP| \cdot |AQ|}$$

Remark. Remember

$$\begin{array}{lll} A & B & P \to Q \\ \downarrow & \uparrow & \\ B & A & P \to Q \end{array}$$

Let A, B be any two points. We denote the hyperbolic line through A and B by l and the ideal endpoints of this line by P and Q. We name those endpoints such that $P * B * A * Q$.

Definition 1.5. The hyperbolic `"distance"` is defined by

$$s(A,B) = \ln(AB, PQ) \tag{1.1}$$

Two segments AB and XY are called `"congruent"` iff $s(A,B) = s(X,Y)$.

Remark. I denote the usual Euclidean distance of A and B by $|AB|$, or $\overline{AB}$ in case a plus or minus value can be meaningfully assigned. But the hyperbolic distance is denoted by $s(A,B)$.

Proposition 1.1 (Additivity of the distance). *Let A, B, C be three points on a hyperbolic line, and assume that B lies between A and C. Then $s(A,B) + s(B,C) = s(A,C)$*

Reason. Again let P and Q be the ideal endpoints of the line through A, B, C. By definition of the hyperbolic distance

$$\begin{aligned}
s(A,B) &= \ln(AB, PQ) = \ln \frac{|AP| \cdot |BQ|}{|BP| \cdot |AQ|} \\
s(B,C) &= \ln(BC, PQ) = \ln \frac{|BP| \cdot |CQ|}{|CP| \cdot |BQ|} \\
s(A,C) &= \ln(AC, PQ) = \ln \frac{|AP| \cdot |CQ|}{|CP| \cdot |AQ|}
\end{aligned} \tag{1.2}$$

Hence

$$\begin{aligned}
s(A,B) + s(B,C) &= \ln(AB, PQ) + \ln(BC, PQ) = \ln\left[(AB, PQ) \cdot (BC, PQ)\right] \\
&= \ln \frac{|AP| \cdot |BQ|}{|BP| \cdot |AQ|} \cdot \frac{|BP| \cdot |CQ|}{|CP| \cdot |BQ|} = \ln \frac{|AP| \cdot |CQ|}{|CP| \cdot |AQ|} \\
&= \ln(AC, PQ) = s(A,C)
\end{aligned} \tag{1.3}$$

as to be shown □

Proposition 1.2 (Distance from the center). *The hyperbolic distance of a point A form the center O is*

$$s(O,A) = 2\tanh^{-1}|OA| \tag{1.4}$$

Proof. One can take for A a point with coordinates $(a,0)$, and B has coordinates $(0,0)$. The ideal endpoints of the horizontal diameter of D are $P = (-1,0)$ and $Q = (1,0)$. Hence the cross ratio is

$$(AO, PQ) = \frac{|AP| \cdot |OQ|}{|OP| \cdot |AQ|} = \frac{(1+a) \cdot 1}{1 \cdot (1-a)}$$

and the hyperbolic distance is

$$s(A,O) = \ln(AO, PQ) = \ln \frac{(1+a) \cdot 1}{1 \cdot (1-a)} = 2\tanh^{-1} a = 2\tanh^{-1}|OA|$$

□

Question. What happens if point A is very near to O? What happens if point A approaches the boundary ∂D?

Answer. If point A is near to the center O, the tangent approximation gives $s(O, A) \simeq 2\,|OA|$. If point A approaches the boundary ∂D, we get $|OA| \to 1$ and hence $s(O, A) \to \infty$. This confirms the Poincaré disk is a model for the unbounded hyperbolic plane.

1.4 The angle of parallelism

This is indeed a remarkable feature of hyperbolic geometry! Here are the basic definitions and facts about the angle of parallelism. Given a line l and a point P not on l, one wants to know what are the parallels to a through P. The asymptotic (or limiting) parallel rays r_+, r_- from vertex P are the two rays that do not intersect line l, but all rays in the interior of the angle $\angle(r_+, r_-)$ do intersect the line l. We drop the perpendicular from P onto line l. Let F be the foot point of that perpendicular and $s = s(P, F)$ be its hyperbolic length.

Definition 1.6 (Angle of parallelism). The angle of parallelism is the angle between either of the asymptotic parallel rays and the perpendicular from P onto line l. The angle of parallelism depends only on the hyperbolic distance s. Following Lobachevskij, one defines a special function, called $\pi(s)$, giving the angle of parallelism π for a segment of hyperbolic length s.

The function $\pi(s)$ is explicitly given by a remarkable formula.

Proposition 1.3 (Lobachevskij's formula for the angle of parallelism). *For any point P and line l, the angle of parallelism $\pi(s)$ relates the hyperbolic distance $s = s(P, F)$ from P to the foot point F of the perpendicular dropped on the line l. Indeed, by the formula*

$$\tan \frac{\pi(s)}{2} = e^{-s} \tag{1.5}$$

Remark. In case one does not assume the axiom of completeness, the quantities on both sides of claim (1.5) are still well defined in the ordered field for segment lengths. *Only solving for s* requires the logarithmic function ln to be defined—to which end one needs the axiom of completeness.

Proof. Let $X_+ = (1, 0), X_- = (-1, 0)$ and $Y_+ = (0, 1), Y_- = (0, -1)$ be the ideal end-points of the horizontal and vertical diameter of D. The hyperbolic distance $s = s(O, P)$ is defined to be

$$s = s(O, P) = \ln(OP, Y_+Y_-) \tag{1.6}$$

in terms of the cross ratio (OP, Y_+Y_-). By definition, this cross ratio is

$$(OP, Y_+Y_-) = \frac{|OY_+| \cdot |PY_-|}{|PY_+| \cdot |OY_-|} = \frac{|PY_-|}{|PY_+|} = \frac{1 + |OP|}{1 - |OP|} \tag{1.7}$$

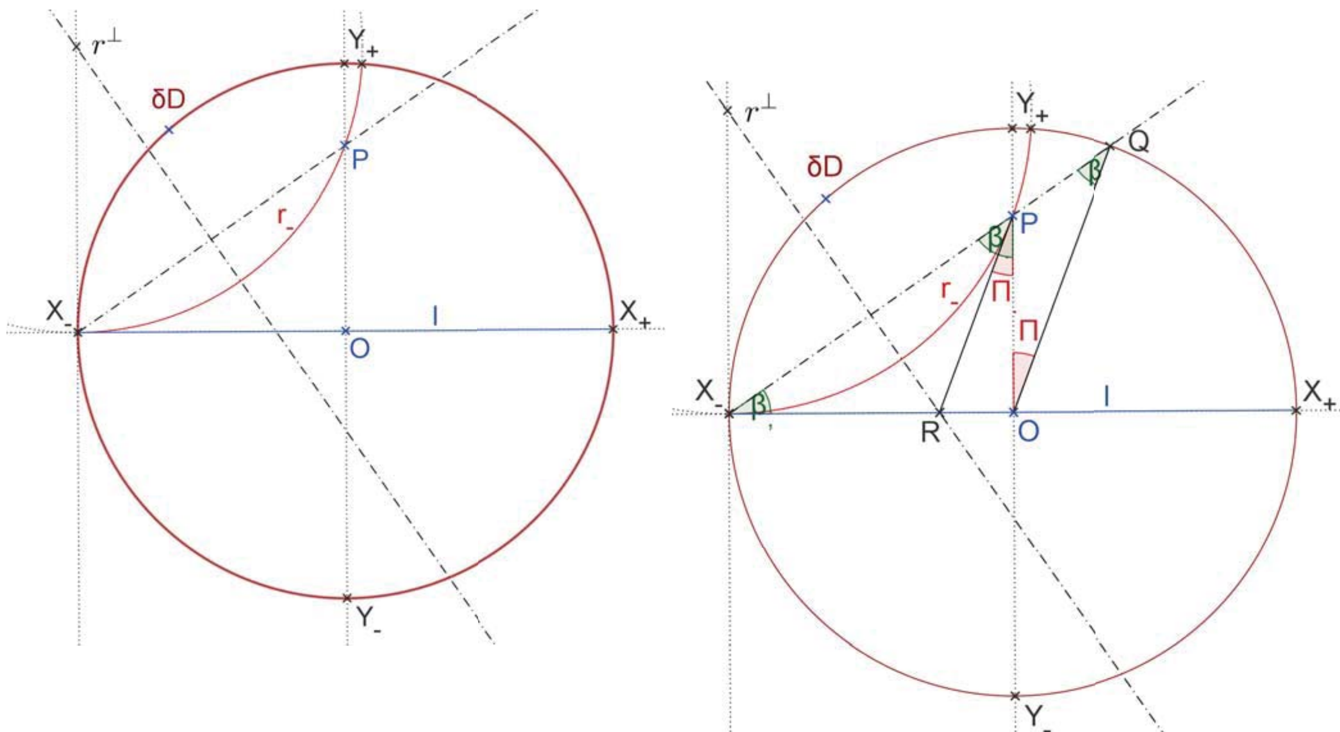

Figure 1.5: How to calculate the angle of parallelism.

Formulas (1.6) and (1.7) allow to calculate the hyperbolic distance $s = s(O, P)$ from the Euclidean distance $y = |OP|$. Indeed

(1.8) $$e^s = \frac{1+y}{1-y}$$

Let R be the intersection point of the horizontal radius X_-O with the Euclidean tangent at point P to the asymptotic ray r_-. The two tangents from point R to the circle r_- form the isosceles $\triangle X_-RP$. Let its congruent base angles be $\beta = \angle RX_-P \cong \angle RPX_-$. The angle sum in the right $\triangle X_-OP$ implies that $\beta + (\beta + \pi(s)) + 90° = 180°$ and hence

(1.9) $$\frac{\pi(s)}{2} = 45° - \beta$$

From the definition of the tangent function from the right $\triangle X_-PO$, one gets

(1.10) $$\tan \beta = |OP| = y$$

One now deducts the final claim from (4)(5) and (6). There are several variants to do that, using more trigonometry, or more geometry. Here is a version relying on trigonometry. We use the addition theorem of tangent and get

$$\tan \frac{\pi(s)}{2} = \tan(45° - \beta) = \frac{\tan 45° - \tan \beta}{1 + \tan 45° \cdot \tan \beta} = \frac{1 - \tan \beta}{1 + \tan \beta}$$

Because of (1.10) and (1.8) we conclude

$$\tan\frac{\pi(s)}{2} = \frac{1-y}{1+y} = e^{-s}$$

as to be shown. $\square$

Problem 1.5. *Given an angle α, state a construction, in the Poincaré model, (and using the underlying Euclidean geometry where needed), to find point P on the segment OY for which $\pi(OP) = \alpha$ is the angle α as given. Actually do an example for this construction!*

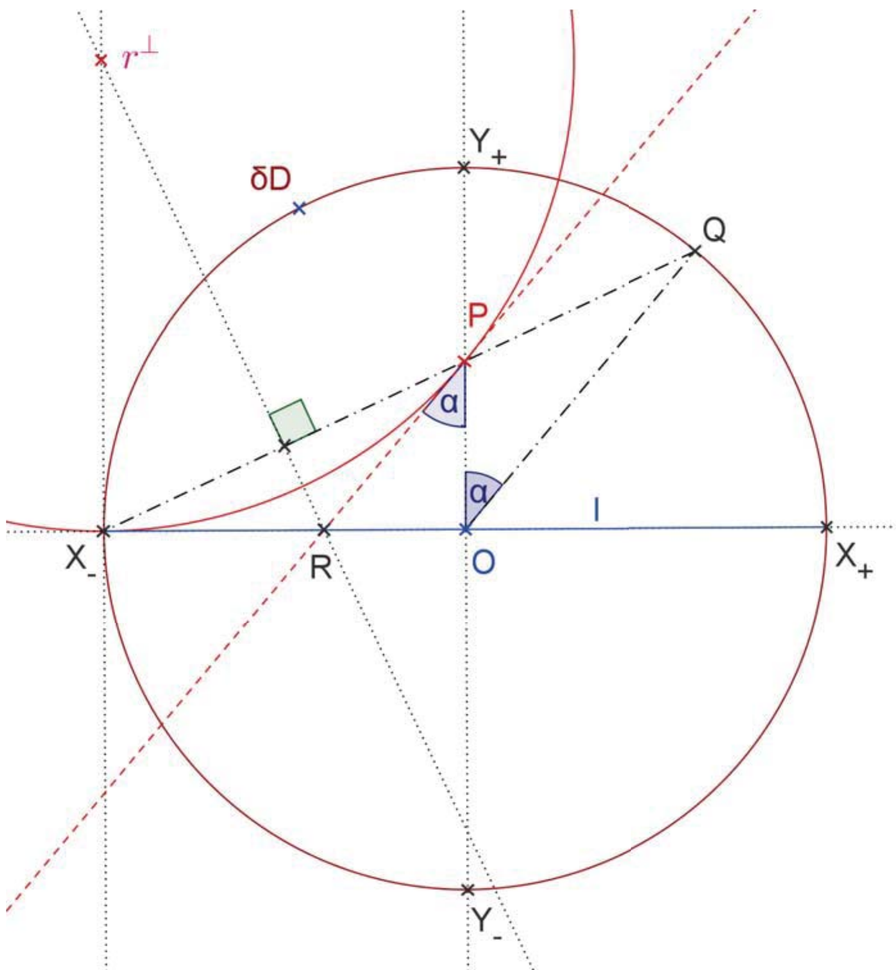

Figure 1.6: For given line l and angle of parallelism $\alpha = 40°$, point P is constructed.

Again, this can be done by putting point P and line l in a suitable special position. We choose the line to be the horizontal diameter of D. Its ideals endpoints are denoted by X_+ and X_-. We choose point P to lie on the positive y-axis. (Indeed, any point and line can be mapped to that special position by a composition of two or three hyperbolic reflections.) In this arrangement, the foot point of the perpendicular from P onto l is the center O. Let Y_+ and Y_- be the ideal endpoints of the perpendicular. The asymptotic parallel rays for point P and line X_+X_- are the hyperbolic rays $r_+ = \overrightarrow{PX_+}$ and $r_- = \overrightarrow{PX_-}$.

Construction 1.3 (**Construction of a segment from given angle of parallelism**). *Let Q be the ideal endpoint in the positive quadrant such that $\angle Y_+OQ = \alpha$. Next draw chord X_-Q and let point P its intersection point with with OY_+. We claim that the segment OP has the angle of parallelism α.*

Reason for validity of the construction. The angle sum in the isosceles $\triangle X'OQ$ implies that $2\beta + (90° + \alpha) = 180°$. By comparison with formula (5) above this implies $\alpha = \pi(OP)$, as claimed. □

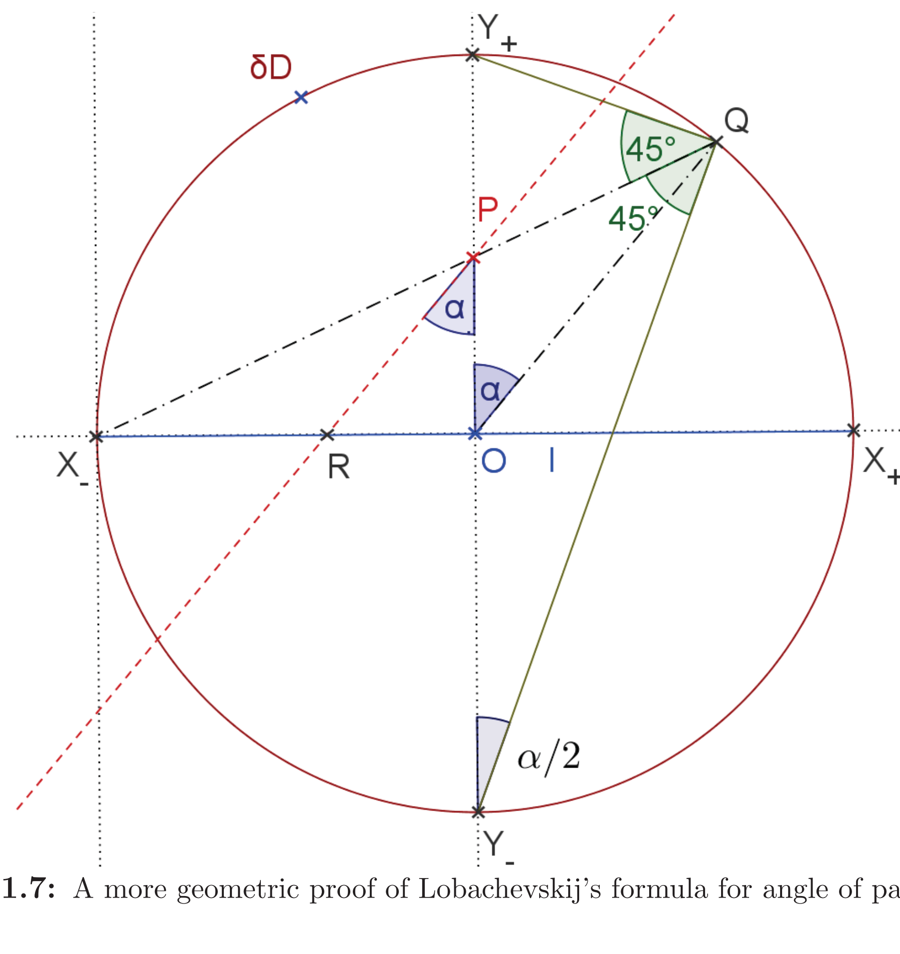

Figure 1.7: A more geometric proof of Lobachevskij's formula for angle of parallelism.

Remark. Too, we get a more geometric proof of Lobachevskij's formula (1.5), avoiding the addition theorem for the tangent function. Repeatedly, we shall use Euclid III.20, and Apollonius' Lemma 1.2 about the angular bisector.

Lemma 1.2. *The angular bisector of any triangle cuts the opposite side in the ratio of the lengths of the two adjacent sides.*

Proposition 1.4 (Euclid III.20). *The angle at the the center of a circle is twice the angle with its vertex at a point of the circumference, and subtending the same arc.*

Proof. By Thales' theorem, $\triangle Y_- Y_+ Q$ is a right triangle. Ray $\overrightarrow{QP}$ is an angular bisector of the right angle. Indeed, by Euclid III.20, $\angle X_- Q Y_- = 45°$, because arc $X_- Y_-$ has the central angle of $90°$. To calculate the ratio in (1.5), we now use the Lemma 1.2 about the angular bisector. Hence

$$\frac{|PY_+|}{|PY_-|} = \frac{|QY_+|}{|QY_-|} \tag{1.11}$$

By Euclid III.20, $\angle Y_+ Y_- Q = \frac{\pi(s)}{2}$, because arc $Y_+ Q$ has the central angle of $\pi(s)$ by construction. Hence the definition of the tangent function implies

$$\tan \frac{\pi(s)}{2} = \frac{|QY_+|}{|QY_-|} = \frac{|PY_+|}{|PY_-|} = \frac{1-y}{1+y}$$

Now we use (1.8) from item 3, and (1.11) above to get Lobachevskij's formula (1.5) once again. □

1.5 Hyperbolic reflection

Proposition 1.5. *The reflection by a hyperbolic line is the same mapping as the inversion by that circle—for the underlying Euclidean plane.*

Proof. Let l be any hyperbolic line and A a point not lying on l. By I_l we denote the inversion by circle l, and the inverted images are denote by subscript l. Note that we study a different inversion here, not the inversion $P \mapsto P'$ by circle ∂D! By definition of inversion, the inverted image A_l of A satisfies

$$|l^\perp A| \cdot |l^\perp A_l| = r_l^2 \tag{1.12}$$

where r_l is the radius of the circular arc l.

To need to show that A_l can be obtained independently by hyperbolic reflection. To this end, we draw the hyperbolic line p through points A and A_l and let S be the intersection of l and p. By definition of reflection, we need to check that

(i) Lines l and p are perpendicular.

(ii) The distances $s(A,S) = s(S,A_l)$ are equal.

The first step is to show that lines l and p intersect perpendicularly. We draw the tangent from the polar $l^\perp$ to circle p, and let T be the touching point of that tangent

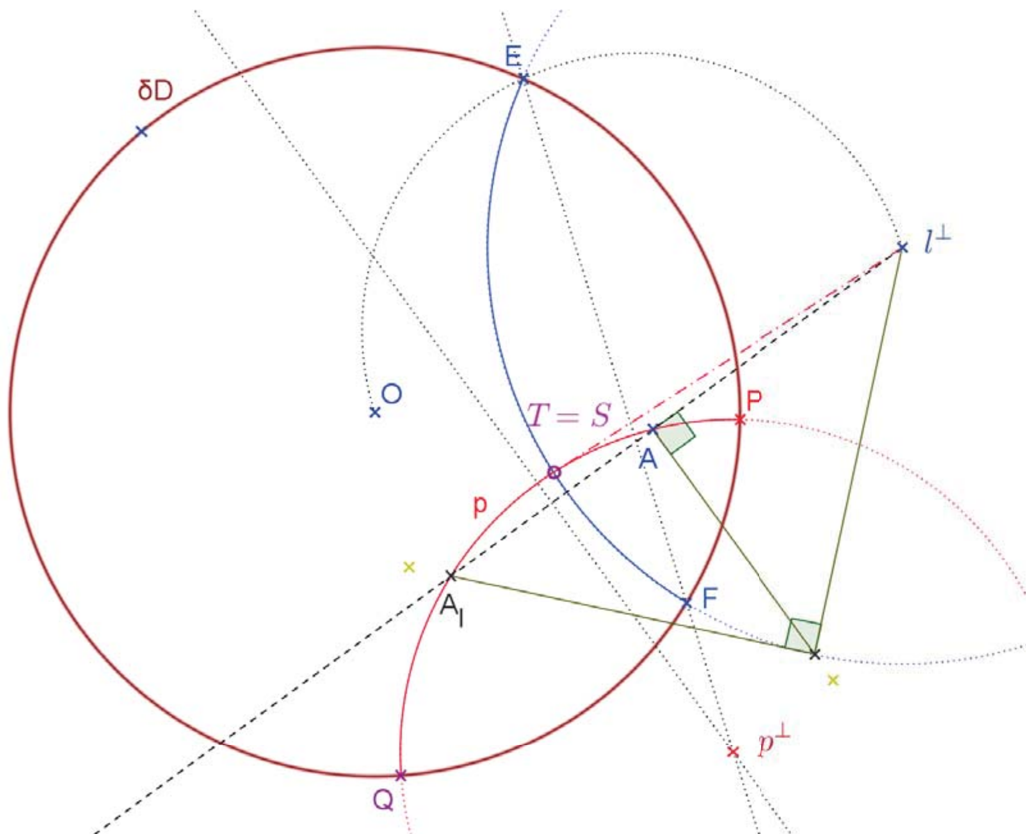

Figure 1.8: The inversion by an orthogonal circle is a hyperbolic reflection.

which lies inside D. By definition, point T lies on circle p. By the theorem of chords (Euclid III.36)

$$|l^\perp A| \cdot |l^\perp A_l| = |l^\perp T|^2 \tag{1.13}$$

Comparison of (1.12) and (1.13) implies that $|l^\perp T| = r_l$. Hence point T lies on circle l, too, and thus $T = S$ is the intersection of circles l and p. The segment $l^\perp T$ is a tangent to circle p as well as a radius of circle l. By Euclid III.16, radius and tangent of circle l are perpendicular to each other. Hence, at the intersection point T, the tangent to circle p is perpendicular to the tangent to circle l.

Next we check item (ii). Any circle perpendicular to l is mapped to itself by the inversion I_l. This follows because inversion maps (generalized) circles to itself and preserves angles. Note that those circles are only mapped to themselves as a set of points, not point by point. Especially, the inversion I_l maps both circles ∂D as well as p to themselves. Hence I_l maps the intersection $p \cap \partial D$ to itself. But $p \cap \partial D = \{P, Q\}$ consists just of the two ideal endpoints of the hyperbolic line p. Because I_l maps the interior of circle l to the exterior, and vice versa, and one of the points P and Q lies in the interior and the other in the exterior of circle l, we conclude that the inversion I_l

maps

$$P \mapsto Q\,,\ Q \mapsto P\,,\ A \mapsto A_l\,,\ A_l \mapsto A\,,\ S \mapsto S \tag{1.14}$$

Especially, this implies that the three points $l^\perp, P, Q$ lie on a Euclidean line.

Proposition 1.6 (**Characterization of perpendicular lines**). *If two hyperbolic lines l and p intersect each other perpendicularly, then the polar $l^\perp$ of one line l and the ideal endpoints P and Q of the other line p lie on a Euclidean line.*

The converse of this statement holds, too. Now we can finally confirm claim (ii): $s(A,S) = s(S,A_l)$. By the definition of hyperbolic distance

$$s(A,S) = \ln(AS, QP)\ ,\ s(S,A_l) = \ln(SA_l, QP) \tag{1.15}$$

The inversion I_l maps points according to (3), and the cross ratio is preserved by inversion. Hence

$$(AS, QP) = (A_l S, PQ) \tag{1.16}$$

As a last step, we use the elementary fact about cross ratios that $(AB, CD) = (BA, DC)$. Hence (1.15) and (1.16) imply $s(A$

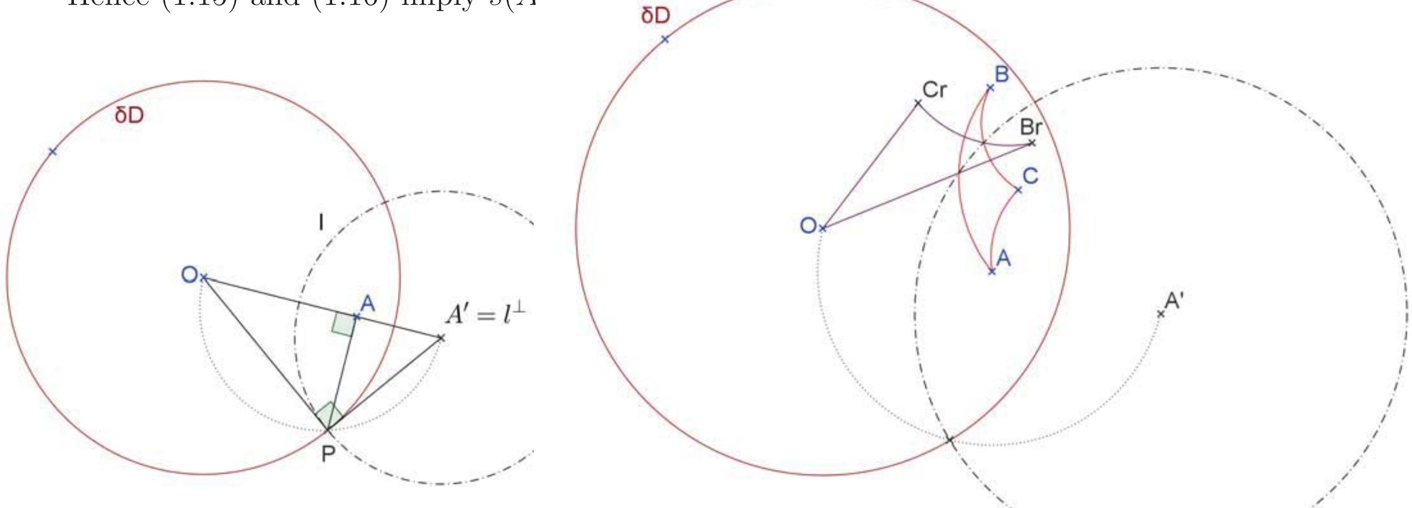

Figure 1.9: The inversion by line l maps the given point A to the center O. Furthermore, B and C are mapped to Br and Cr.

Proposition 1.7 (**An especially useful reflection**). *For a given point A, there exists a unique the hyperbolic line l, the inversion by which maps A to the center O.*

Problem 1.6. *Explain a construction for this hyperbolic line l.*

Construction. Draw the radial ray $\overrightarrow{OA}$. Erect on it the perpendicular at point A. Let P be one of the two points where the perpendicular intersects the circle ∂D. Erect the perpendicular on the radius OP. This is a tangent to the circle ∂D. It intersects the ray $\overrightarrow{OA}$ in the inverse point A'. The polar of the reflection line is $l^\perp = A'$. The hyperbolic line l is given, in the Poincaré model, by the circle with center $l^\perp$ through point P. $\square$

1.6 Proof of the SAS axiom via the Poincaré model

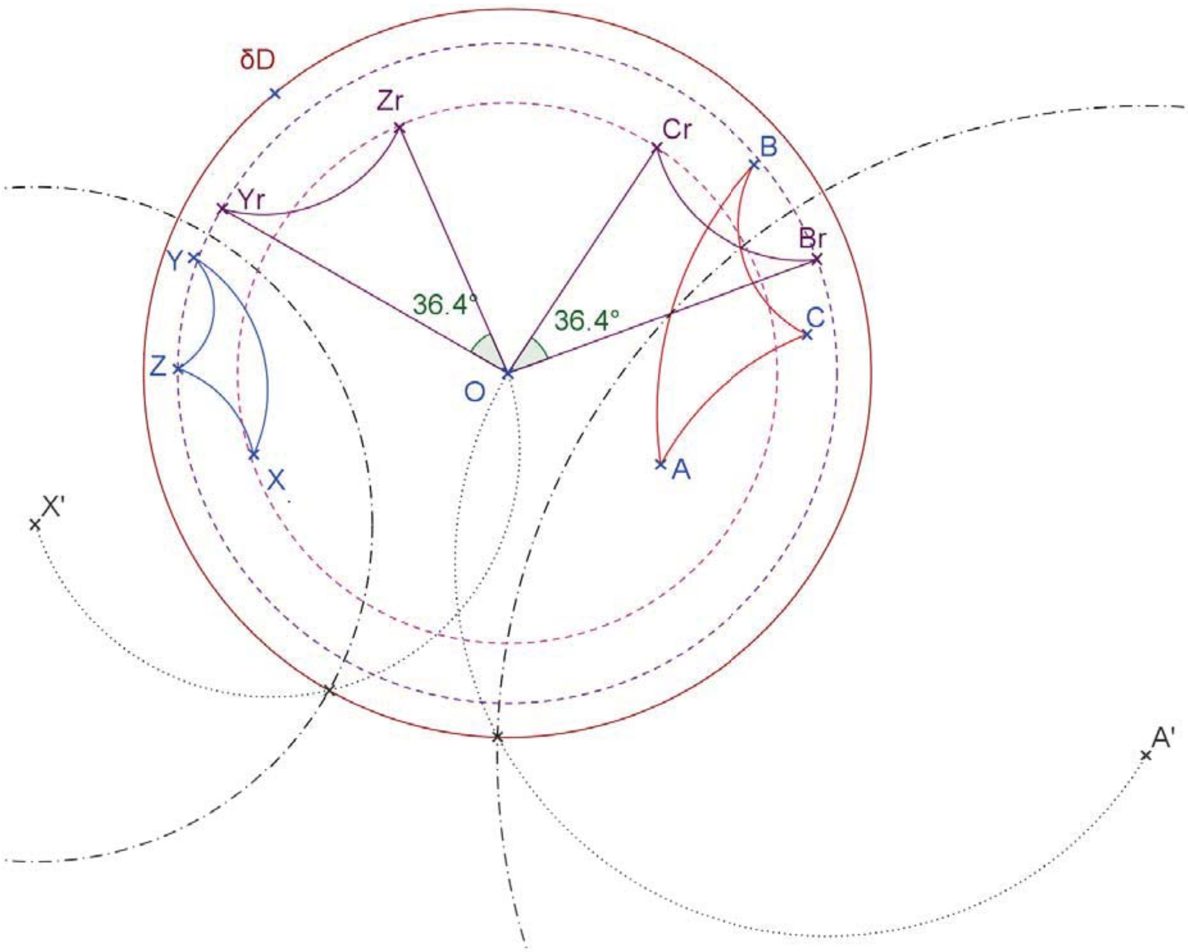

Figure 1.10: Sorry, the triangles $\triangle ABC$ and $\triangle XYZ$ are not congruent.

Proposition 1.8 (SAS congruence for the Poincaré Model). *Given are two triangles $\triangle ABC$ and $\triangle XYZ$, with the angles at vertices A and X, and adjacent sides pairwise congruent:*

$$\angle CAB \cong \angle ZXY\ ,\ s(A,B) = s(X,Y)\ ,\ s(A,C) = s(X,Z) \tag{1.17}$$

Then the two triangles are congruent.

Reason. By construction (a) above, there exists a hyperbolic line α, the inversion I_α by which maps A to O. In the same way, there exists a hyperbolic line ϕ such that the inversion I_ϕ maps point X to the center O. We map $\triangle ABC$ by the inversion I_α to get the congruent $\triangle OB_\alpha C_\alpha \cong \triangle ABC$. Similarly, we map $\triangle XYZ$ by the inversion I_ϕ

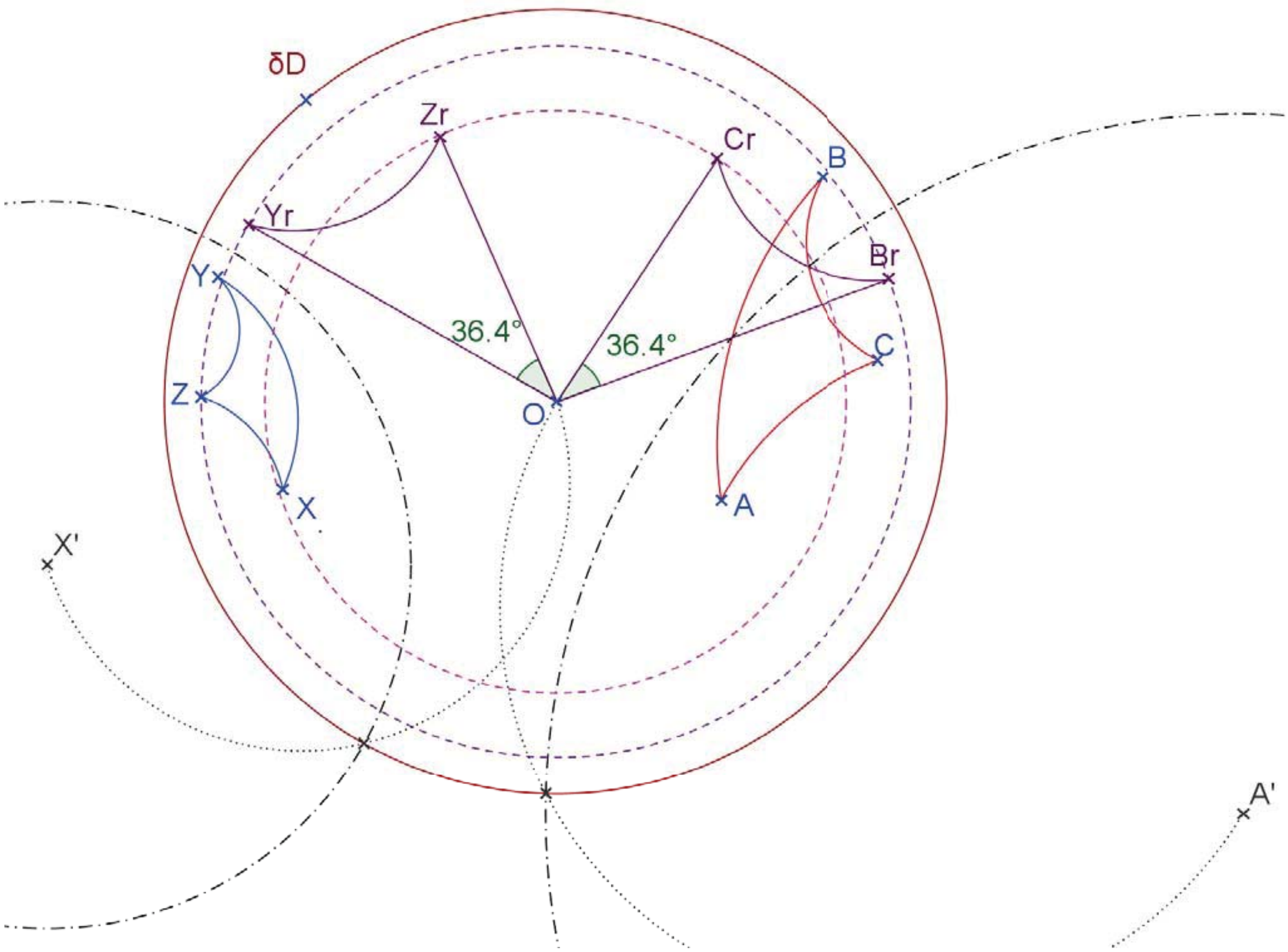

Figure 1.11: SAS congruence.

to get the congruent $\triangle OY_\phi Z_\phi \cong \triangle XYZ$. Moreover, assumption (1) and transitivity imply

$$\angle C_\alpha OB_\alpha \cong \angle Z_\phi OY_\phi\,,\ s(O,B_\alpha) = s(O,Y_\phi)\,,\ s(O,C_\alpha) = s(O,Z_\phi) \tag{1.18}$$

Now it is quite easy to show that the two triangles $\triangle OB_\alpha C_\alpha$ and $\triangle OY_\phi Z_\phi$ are congruent (still in the hyperbolic sense!). Indeed, lines through the center O are lines both in the Euclidean and hyperbolic sense. The usual Euclidean reflection across such a line is a hyperbolic reflection, too. One such reflection will map ray $\overrightarrow{OB_\alpha}$ to ray $\overrightarrow{OY_\phi}$, and $\triangle OB_\alpha C_\alpha$ to $\triangle OB'C'$. Hence one gets a hyperbolic congruence

$$\triangle OB'C' \cong \triangle OB_\alpha C_\alpha \cong \triangle ABC \tag{1.19}$$

and the rays $\overrightarrow{OB'} = \overrightarrow{OY_\phi}$ are equal. In case that points C' and Z_ϕ lie on different sides of ray OB_α, one maps $\triangle XZY$ by a reflection across that ray to get a hyperbolic congruence

$$\triangle OY'Z' \cong \triangle OY_\phi Z_\phi \cong \triangle XZY \tag{1.20}$$

In case that points C' and Z_ϕ lie on the same side of ray OB_α, one needs no second reflection, but puts $\triangle OY'Z' := \triangle OY_\phi Z_\phi$, and one gets formula (1.20) once more.

Now, after those further mappings by reflections, the triangles $\triangle OB'C'$ and $\triangle OY'Z'$ are very easy to compare. Indeed (1.18) implies

$$\angle C'OB' \cong \angle Z'OY'\,,\ s(O,B') = s(O,Y')\,,\ s(O,C') = s(O,Z') \tag{1.21}$$

and the rays $\overrightarrow{OB'} = \overrightarrow{OY'}$ are equal, and points C' and Z' lie on the same side of that ray. everybody can check that this simply implies that $B' = Y'$ and $C' = Z'$. (Actually, one uses the uniqueness for the lay off of angles and segments in Euclidean geometry.) Now (1.19) and (1.20) and transitivity imply that the hyperbolic congruence $\triangle ABC \cong \triangle XYZ$ as to be shown. □

1.7 Their are enough rigid motions

Theorem 1.1 (There are enough rigid motions). *Given are two points P and Q, and rays r_P and r_Q with these vertices. There exist exactly* <u>two</u> *rigid motions, which map P to Q and r_P to r_Q.*

Proof of the Theorem about enough rigid motions. As explained in proposition 1.7, there exist an inversion I_l which maps the given point P to the center O, and, similarly, an inversion I_k which maps the given point Q to the center O. There exists a reflection I_m by a line m through O which maps the ray $I_l(r_P)$ to the ray $I_k(r_Q)$. The composition $I_k \circ I_m \circ I_l$ of these three mappings is a rigid motion which maps point P to point Q and ray r_P to ray r_Q.

The second rigid motion which maps point P to point Q and ray r_P to ray r_Q, too, is given by the composition $I_k \circ I_m \circ I_l \circ I_r$ where I_r is the reflection across the line of r_P.

From the lemma below, we see that these two mappings are the only rigid motions with the required properties. □

Lemma 1.3. *Given any point P and ray r_P. There exist exactly* <u>two</u> *rigid motions, which leave the point P and the ray r_P fixed. One of them is the identity, the other one is the reflection I_r across the line of r_P.*

Proof. Let ϕ be a rigid motion leaving the point P and the ray r_P fixed. As easily checked, ϕ leaves all points on the line of r_P fixed. Furthermore, the mapping either exchanges the halfplanes of this line or leaves them invariant.

In the first case, the mapping ϕ is the identity. This is an easy consequence of the unique transfer of segments and angles.

Consider the second case that the mapping ϕ exchanges the halfplanes. Let I_r be the reflection across the line of r_P, and define the new mapping $\psi := \phi \circ I_r$. The mapping ψ leaves the point P and the ray r_P fixed, and leaves the halfplanes invariant. Hence, by the reasoning above, ψ is the identity and hence $\phi = I_r$. $\square$

The following explanation is almost unnecessary, as long as we set up the hyperbolic geometry only in the Poincaré model:

Given is any point A, and ideal point (endpoint) E. Let AE denote the line through A with endpoint E. For any line l, one ray of which is a limiting parallel to AE, we simply say that l "goes through the ideal point E".

Corollary 57. *Let any two points P and Q, and two ends E and F be given. There exist exactly* <u>*two*</u> *rigid motions, which map P to Q and the ray PE to the ray QF.*

Corollary 58. *Every hyperbolic rigid motion of the Poincaré disk can be realized by a composition of* $0, 1, 2, 3$ *or* 4 *hyperbolic reflections.*

Corollary 59. *Realizing these hyperbolic reflections with inversion by circles or lines produces a unique extension of any rigid motion as a bijective mapping of the underlying Euclidean plane. The extended rigid motion conserves the pairs of inverse points.*

Proof. The inversion I_l which realizes the hyperbolic reflection by the line l has the following property: If the inversion I_l maps point A to point B, then it maps the inverse point A' to the inverse point B'—or written in one formula:

$$[I_l(A)]' = I_l(A')$$

To check this conjecture, take any two hyperbolic lines c and d intersecting in point A. The corresponding circles in the Poincaré disk model have the inverse image A' as their second intersection point.

The inversion I_l maps c and d into two circles $I_l(c)$ and $I_l(d)$, with the intersection points $I_l(A)$ and $I_l(A')$. Since both $I_l(c)$ and $I_l(d)$ are orthogonal to the line of infinity δD, they consist of pairs of inverse points. Hence $[I_l(A)]' \in I_l(c) \cap I_l(d) = \{I_l(A), I_l(A')\}$ and $[I_l(A)]' = I_l(A')$ as claimed.

Every hyperbolic rigid motion of the Poincaré disk can be realized by a composition ϕ of up to four inversions by hyperbolic lines. Since

$$[\phi(A)]' = \phi(A')$$

holds for any such composition, too, we see that the extended rigid motion conserve pairs of inverse points, too. $\square$

Proposition 1.9. *Given a hyperbolic line l and two points A and B symmetric to this line, any rigid motion ϕ maps these objects to an image line and two image points symmetric with respect to the image line. Hence*

$$\phi(I_l(A)) = I_{\phi(l)}(\phi(A)) \quad \text{and} \quad I_{\phi(l)} = \phi \circ I_l \circ \phi^{-1}$$

Theorem 1.2. *Given any three different ends E, F and G, as well as their images three different ends E', F' and G'. exactly one rigid motion, which maps E to E', F to F', and G to G',*

1.8 Horocycle

Problem 1.7. *For the following propositions and theorem, provide drawings in the Poincaré disk model, using compass and straightedge. Copy, use and complete the drawings to prove the statements.*

Definition 1.7 (horocycle). A horocycle $\mathcal{H}$ around the endpoint E through point A consists of all points A_l obtained from A by a reflection across any line l through the endpoint E.

Problem 1.8. *Explain why a rigid motion maps a horocycle bijectively onto a horocycle.*

Lemma 1.4. *In the Poincaré disk, the horocycle around the ideal point E through the center O is depicted by the circle with diameter OE.*

Proof. In the figure on page 771, the center O is reflected across the lines l and k with the common endpoint E. As shown in proposition 1.5, the hyperbolic reflection is an inversion by an orthogonal circle.

We now argue using the underlying Euclidean plane. and use the construction 1.1 to obtain the inverse point O_l. [55] The inverted point O_l is midpoint of the chord between the end E and F of arc l. Too, it is the intersection of the ray $Ol^\perp$ with the chord between the end E and F of arc l, and we get a right angle $\angle OO_lE$. By the converse Thales' theorem, we see that Ol lies on a circle with diameter OE. Hence points of the horocycle around the ideal point E through the center O, as given by definition 1.7, lie on the circle with diameter OE. As easily seen, the converse holds, too. □

Lemma 1.5. *In the Poincaré disk, the horocycle around the ideal point E through any point A is depicted by the circle through A touching the line of infinity δD at the ideal point E.*

Proof. By theorem 1.1, there exists a rigid motion ϕ which maps point A to the center O, but leaves the end E fixed. The horocycle $\mathcal{H}$ through O is depicted as a circle through O touching the line of infinity ∂D from inside at the ideal point E. The inverse image $\phi^{-1}(\mathcal{H})$ is depicted as a circle through A touching the line of infinity ∂D from inside at the ideal point E. Clearly the inverse image is a horocycle, too. □

Definition 1.8 (tangent to a horocycle). Given is the horocycle $\mathcal{H}$ around the endpoint E through point A. The line t through A perpendicular to AE is called the *tangent* to the horocycle at point A.

[55] This time we go from outside to inside the circle l.

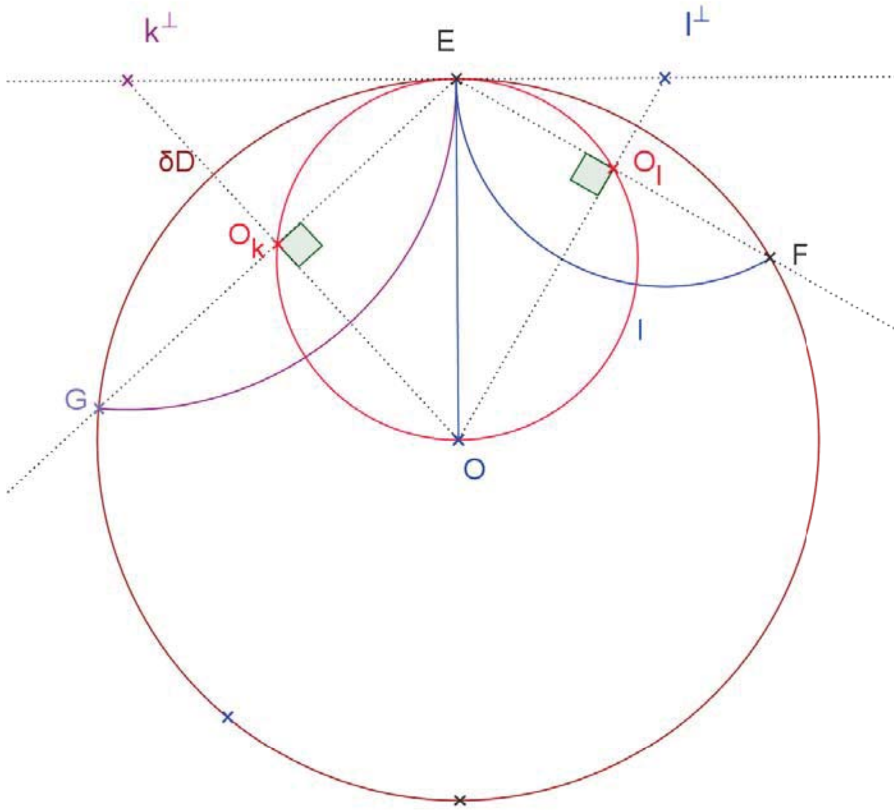

Figure 1.12: The set of the reflective images of center O across all lines with endpoint E yields the horocycle around E through O.

Proposition 1.10. *The tangent to a horocycle at any point B meets the horocycle only at this touching point. Any other line through the touching point cuts the horocycle in two points $B \neq C$.*

Proof. There exists a rigid motion ϕ which maps point B to the center O. Hence it is enough to prove the claim for a horocycle $\mathcal{H}$ through O. Let E be the end around which this horocycle goes. In the Poincaré model $\mathcal{H}$ is depicted as a circle with diameter OE. The tangent to $\mathcal{H}$ at O is depicted as the diameter of ∂D perpendicular to OE. Any other hyperbolic line through O is depicted as another diameter of ∂D. Hence it intersects $\mathcal{H}$ in two points—which clearly are hyperbolic points, as to be shown. $\square$

Lemma 1.6. *For horocycle around the ideal point E through the center O of the Poincaré disk, and any line l with ideal point E, the limit triangle $\triangle OEO_l$ has congruent angles at vertices O and O_l.*

Proof. By definition of the horocycle, these two points O and

$$Q = I_l(O)$$

are mirror images for a hyperbolic reflection across a line l with one end E. There exists a rigid motion ψ which maps line l to a diameter $\psi(l)$ and leaves E fixed. As image of the mapping, we get the horocycle $\psi(\mathcal{H})$ around E through the two points $\psi(O)$ and $\psi(Q)$. As explained above,

$$\psi(Q) = I_{\psi(l)}(\psi(O))$$

and the two image points are symmetric mirror images by the line $\psi(l)$, Since $\psi(l)$ is a diameter of δD, this holds in the Euclidean sense, too. And since $E = \psi(E)$, and this point is fixed both by the inversion I_l and the inversion $\psi(l)$, it lies on the diameter $\psi(l)$. Hence the limit triangle $\triangle\psi(O)\psi(E)\psi(Q)$ is symmetric to this diameter both in the Euclidean, and the hyperbolic sense. Hence it has congruent base angles, and the limit triangle $\triangle OEQ$ has congruent base angles, too, as to be shown. □

Lemma 1.7. *If point B lies on the horocycle around A through the ideal end E, and point C lies on the horocycle around B through the same ideal end E, then point C lies on the horocycle around A through the ideal end E.*

Proof. Assume point B lies on the horocycle $\mathcal{H}_A$ around the ideal end E. Assume point C lies on the horocycle $\mathcal{H}_B$ around B through the same ideal end E,

In the Poincareé model, horocycle $\mathcal{H}_A$ is depicted as a circle touching the line of infinity δD at the ideal point E from inside, and going through point A. A circle is uniquely determined by two points, and the tangent at one of them. Hence $\mathcal{H}_A = \mathcal{H}_B$. From this, we see that point C lies on this same horocycle, too. □

Proposition 1.11. *For a horocycle around the ideal end E through any two points B and C, the limit triangle $\triangle BCE$ has congruent angles at vertices B and C.*

Proof. There exists a rigid motion ϕ which maps point B to the center O. Let $\phi(C) = Q$. It is enough to prove the claim for the image horocycle $\phi(\mathcal{H})$ through O and Q. But this has already been done in the Lemma above. □

Proposition 1.12. *Given any point A and ideal point E. The set of all points B such that the limit triangle $\triangle ABE$ has congruent angles at vertices A and B is equal to the horocycle around the ideal point E through point A.*

1.9 Circles and hypercircles

Recall the easy definitions from neutral triangle geometry.

Definition 1.9 (Circle). Given is a center A and a distance AX. The set of all points with distance from the center O congruent to AX is called a *circle*.

Definition 1.10 (Equidistance line). Given is a baseline l and a distance AX. The set of all points with distance from a baseline l congruent to AX, and lying on one side of this line, are called an *equidistance line* or *hypercycle*.

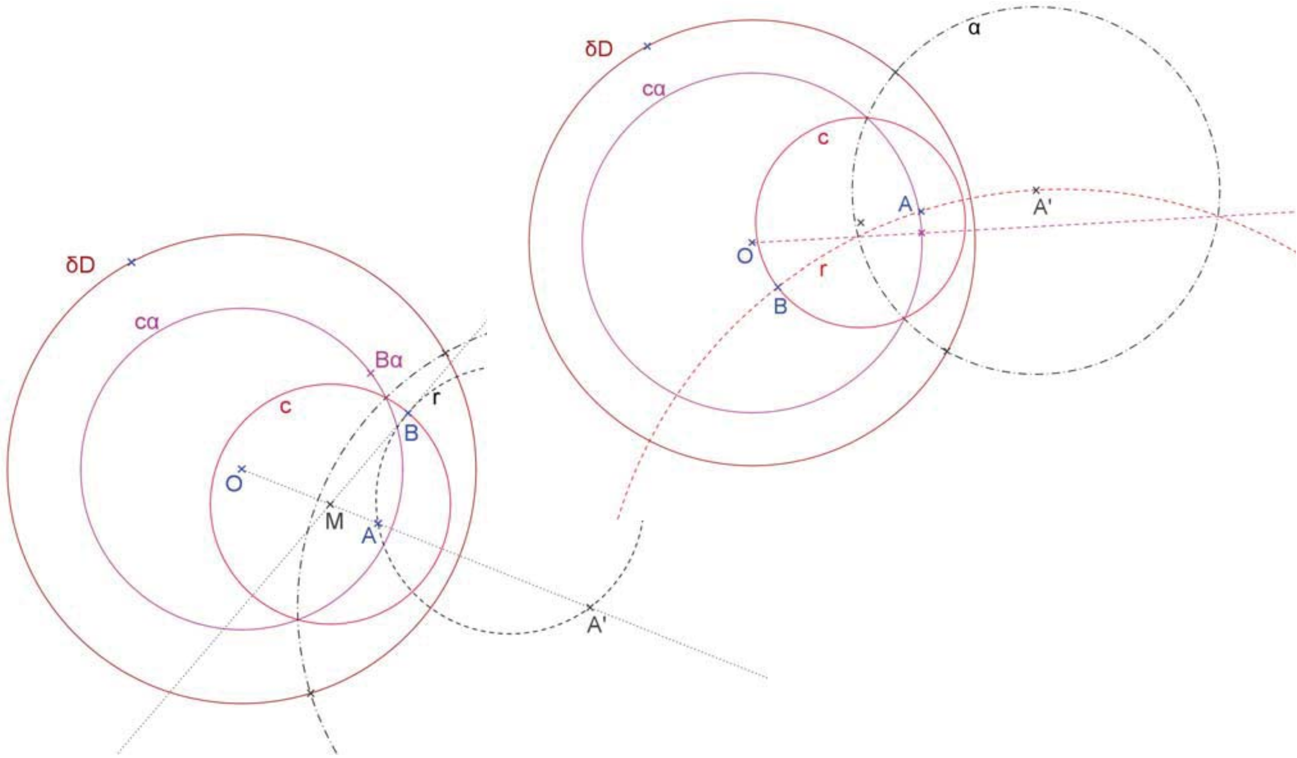

Figure 1.13: A circle appears as a circle—evidently for a circle about O, and hence always by means of a useful reflection.

Proposition 1.13 (Circles appear as circles). *In the Poincaré disk model, a circle appears as a circle* inside *the disk D.*

Reason. In the special case that the center of the given circle $A = O$ is the center of the Poincaré disk, the statement is obviously true. Otherwise, we use Proposition 1.7. There exists a unique orthogonal circle l, the inversion by which maps the center A of the given circle $\mathcal{A}$ to the center O of the Poincaré disk.

By Proposition 1.5, the inversion by l is a hyperbolic reflection. Hence the circle $\mathcal{A}$ is mapped to a circle around O, which appears in the Poincaé model as a circle $\mathcal{C}$ around O.

A second application of the inversion by l maps circle $\mathcal{C}$ back to the original given circle $\mathcal{A}$. As we have shown in the section about inversion by circles, inversion maps circles to circles. Hence $\mathcal{A}$ appears as a circle inside Poincaré's model. □

Proposition 1.14 (Equidistance lines are circular arcs). *In the Poincaré disk model, the set of all points of a given hyperbolic distance d from a given line l lie on two circular arcs.*

Reason. It suffices to take for the line l the horizontal diameter EF of ∂D, and choose a point B on the vertical diameter m. The foot point of the perpendicular from B onto l is $A = O$. By definition, the hyperbolic distance of the two points A and B is

$$s(A, B) = \ln(AB, PQ) \tag{1.22}$$

where P and Q are the ideal endpoints of the vertical diameter m.

Let ε be the circular arc through point B and the two ideal endpoints E, F of line l. Note that the circular ε is neither a line nor a circle in hyperbolic geometry!

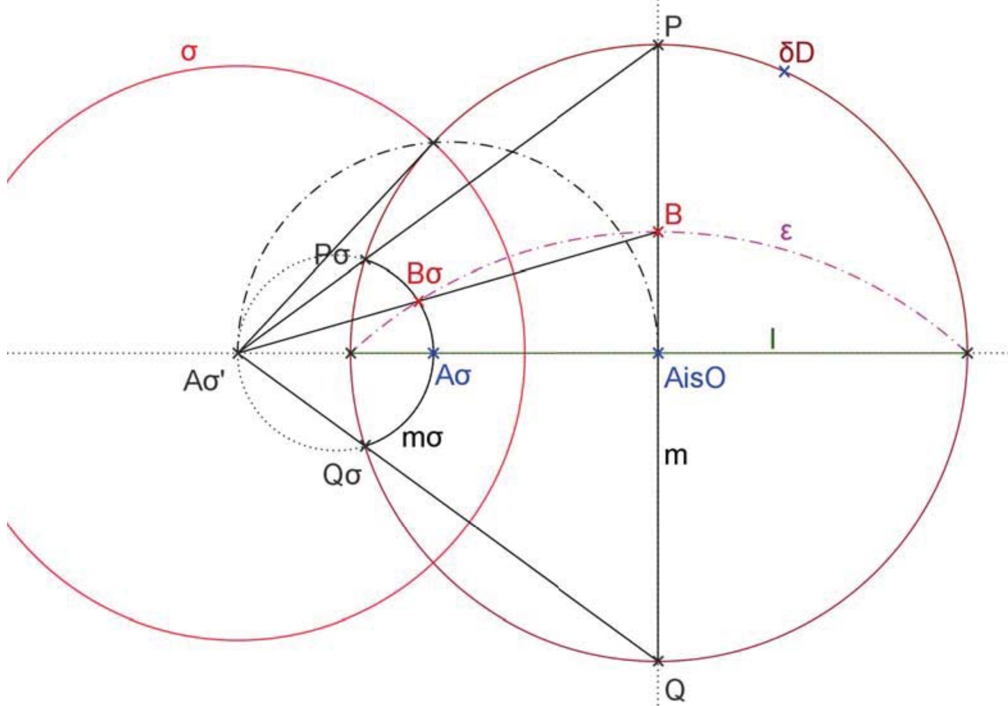

Figure 1.14: An equidistance line is a circular arc with two ideal endpoints.

Question. Why is ε not a hyperbolic line?

Answer. Circles ∂D and ε do not intersect perpendicularly.

We need to show that all points of ε have the same distance from line l. One begins by choosing an arbitrary point B_σ of ε. Let $\sigma^\perp$ be the intersection of the Euclidean lines l and BB_σ. Construct the tangent from $\sigma^\perp$ to the circular arc ε, and let S be the touching point of the tangent. Let σ be a circular arc through S around $\sigma^\perp$.

Is σ a hyperbolic line? To answer that question, apply Euclid III.36 to circle ε and chords through $\sigma^\perp$. One gets

$$|\sigma^\perp S|^2 = |\sigma^\perp B| \cdot |\sigma^\perp B_\sigma| = |\sigma^\perp|E \cdot |\sigma^\perp F| \tag{1.23}$$

This shows that by inversion I_σ across the circular arc σ maps:

$$S \mapsto S,\ B \mapsto B_\sigma,\ E \mapsto F,\ F \mapsto E \tag{1.24}$$

The inversion I_σ maps line l to itself. Hence by preservation of angles and generalized circles, the inversion I_σ maps circle ∂D to itself. Is circle ∂D mapped to itself point by point?

Answer. No, formula (1.24) shows that the two points E, F of circle ∂D are interchanged by the mapping I_σ.

Because inversion by σ maps the entire circle ∂D to itself, circles ∂D and σ intersect perpendicular, and hence σ is a hyperbolic line. Hence, as shown in the lecture, inversion by circle σ is a hyperbolic reflection. Next we map the points A, P and Q of m by this reflection, and get the reflected points A_σ, P_σ and Q_σ. The two points P_σ and Q_σ are the ideal endpoints of line of the reflected line m_σ. Because line m_σ and l are perpendicular to each other, the perpendicular from B_σ onto line l has the foot point A_σ, and hence $d(A_\sigma, B_\sigma)$ is the distance form point B_σ to line l. Because the cross ratio is conserved by circular inversion, we get

$$s(A_\sigma, B_\sigma) = \ln(A_\sigma B_\sigma, P_\sigma Q_\sigma) = \ln(AB, PQ) = s(A, B)$$

This shows that points B and B_σ have the same distance from line l. Because point B_σ was chosen arbitrarily, we conclude that all points of the arc ε have the same distance from line l. The second arc the points of which have the same distance from l is produced by reflection across line l. $\square$

1.10 We have obtained all circle-like curves

It can happen in hyperbolic geometry that *three points lie neither on a line nor a circle.* We have seen three types of circle-type curves: circle, equidistance line, and horocycle. We now obtain—via the Poincareḿodel—a satisfactory theorem confirming that we have found all types of circle-type curves.

Theorem 1.3. *In the hyperbolic plane, any three different points lie either on a line, a circle, a horocycle, or an equidistant line. In the Poincaredisk model*

a ***circle*** *is depicted as a circle lying inside the disk D.*

a ***hyperbolic line*** *is a circular or straight arc, which has two intersection points with the line of infinity ∂D, and intersects it perpendicularly.*

an ***equidistance line*** *is a circular or straight arc, which has two intersection points with the line of infinity ∂D, but not a perpendicular angle of intersection.*

a ***horocycle*** *is a circle touching the line of infinity δD from inside This is simply the remaining possible case!*

1.11 Towards the Klein model

The relation of hyperbolic lines and their chords is useful for the translation from Poincaré's to Klein's model.

Proposition 1.15 (Hyperbolic lines and their chords). *Given are two or more hyperbolic lines $l, l_1, l_2, \ldots$ all through one point P.*

(a) *The chords between the ideal endpoints of lines $l, l_1, l_2 \ldots$ all intersect in one common point K, too.*

(b) *The inverse point K' is the midpoint of the segment PP' between P and its inverse point P'.*

(c) *The location of the point K on the ray $\overrightarrow{OP}$ is given by*

$$|OK| = \frac{2\,|OP|}{1 + |OP|^2} \tag{1.25}$$

For the proof, we need to recall, from Euclidean geometry of circles II, the definition of *power* $p(O, \mathcal{C})$ of a point O with respect to a circle $\mathcal{C}$: For any circle $\mathcal{C}$ and point O, let

$$p(O, \mathcal{C}) = |OA| \cdot |OB|$$

where A and B are the two intersection points of a line l through O with the circle $\mathcal{C}$. If a second line k intersects the circle $\mathcal{C}$ in the points P and Q, by Euclid III.35 and III.36,

$$|OA| \cdot |OB| = |OP| \cdot |OQ|$$

Hence the power p does not depend on the choice of the line l, and thus is well defined. The power is negative for points inside the circle and positive for points outside the circle.

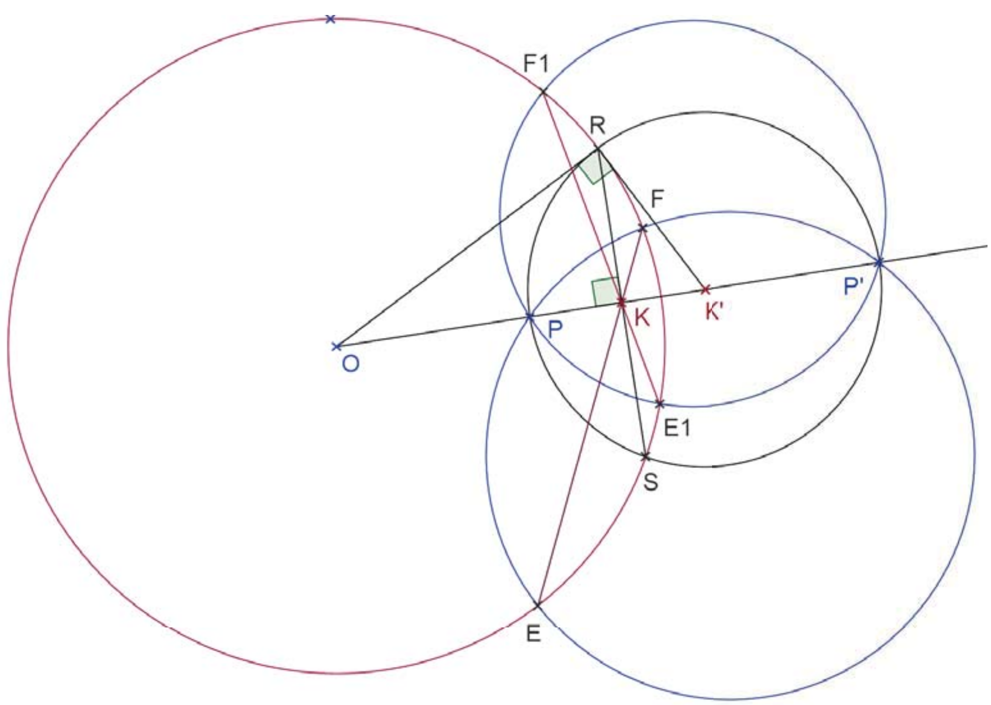

Figure 1.15: The chords of a bundle of hyperbolic lines through a common point P intersect in a common point K.

Proof of (a). Pick one line l, let E, F be the ideal endpoints of l, and define point K as the intersection of chord EF with chord PP', where P' is the inverse image of P. Actually, EF is the common chord of ∂D and l, and hence

$$p(K, \partial D) = |KE| \cdot |KF| = p(K, l)$$

We now calculate the power of K relative to l. Since PP' is the common chord of all hyperbolic lines through point P.

$$p(K, l) = |KE| \cdot |KF| = |KP| \cdot |KP'|$$

Clearly the last two formulas imply

$$p(K, \partial D) = |KP| \cdot |KP'|$$

which confirms that $p(K, \partial D)$ is independent of the choice of the line l. Since point K lies on the ray $\overrightarrow{OP}$, we conclude point K is independent of the choice of the line l.

(We may repeat the same process for another line l_1. It would be possible that we get another intersection point K_1 of chord E_1F_1 with chord PP'. But, as we have shown, $K_1 = K$ does hold.) □

Proof of (b). We draw the circle t with diameter PP'. Let K' be its center, and RS be the common chord of circles ∂D and t. Since t passed through the pair of inverse points P and P', the circles t and ∂D are orthogonal to each other. Hence $\angle ORK' = 90°$ and RK is the altitude of the right $\triangle ORK'$. Now $|OK| \cdot |OK'| = |OR|^2 = 1$ follows from the leg theorem in this $\triangle OSK'$. Hence the center K' of circle t is the inverted point of the foot point K. By construction, K' is the midpoint of diameter PP'. □

Proof of (c). Since the inverse point K' is the midpoint of the segment PP' between P and its inverse point P', we use the definition of inverse points twice and get

$$|OK| = \frac{1}{|OK'|} = \frac{2}{|OP'| + |OP|} = \frac{2\,|OP|}{|OP'| \cdot |OP| + |OP|^2} = \frac{2\,|OP|}{1 + |OP|^2}$$

which confirms claim (1.25). □

2 Geometric Constructions in the Poincaré Disk

Before we start. These constructions are done using compass and straightedge of the *ambient Euclidean plane* and are to be described in terms of the Euclidean geometry of that ambient plane. Use the conventions from the section about Poincaré's model to keep your explanations short! For each problem, provide drawings and set up a step by step construction process.

2.1 Basic constructions from neutral geometry

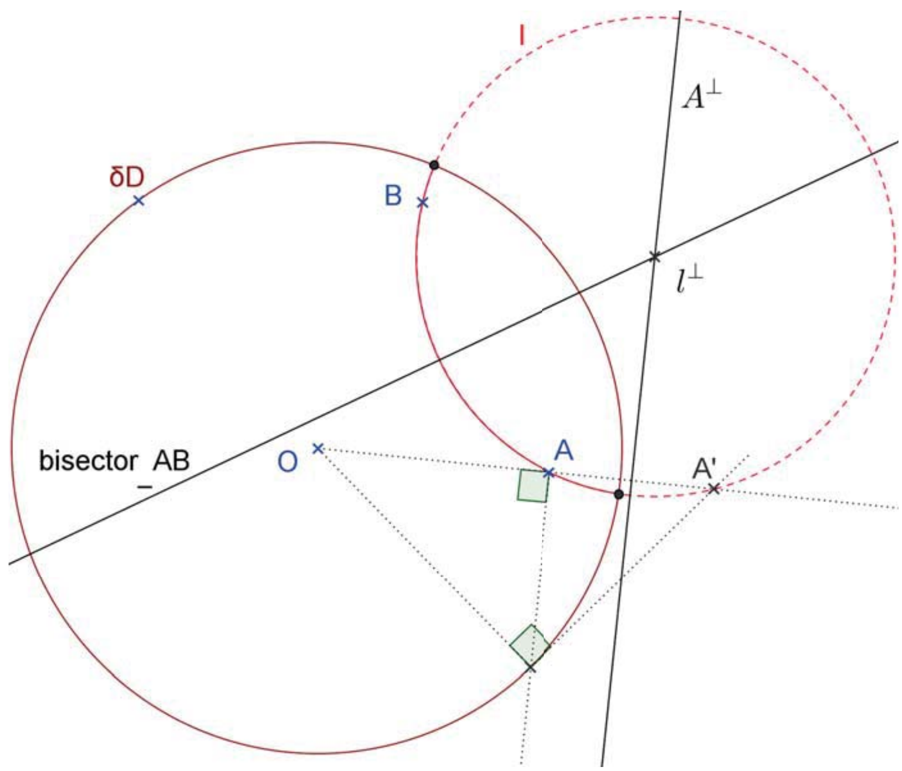

Figure 2.1: How to construct a hyperbolic line through two given points A and B.

Problem 2.1. *For two given points A and B, construct the hyperbolic line l through A and B.*

Construction 2.1. *One needs the inverse point of either A or B, say A'. The hyperbolic line l is modelled by a circle through A, A' and B. Its center $l^\perp$ is found as intersection point of the perpendicular bisectors of the sides of $\triangle AA'B$. One can, for example construct the perpendicular bisector $A^\perp$ of segment AA', and the perpendicular bisector of AB. Finally, one draws a circular arc around $l^\perp$ through point A. This arc passes through points B, A' and B', too.*

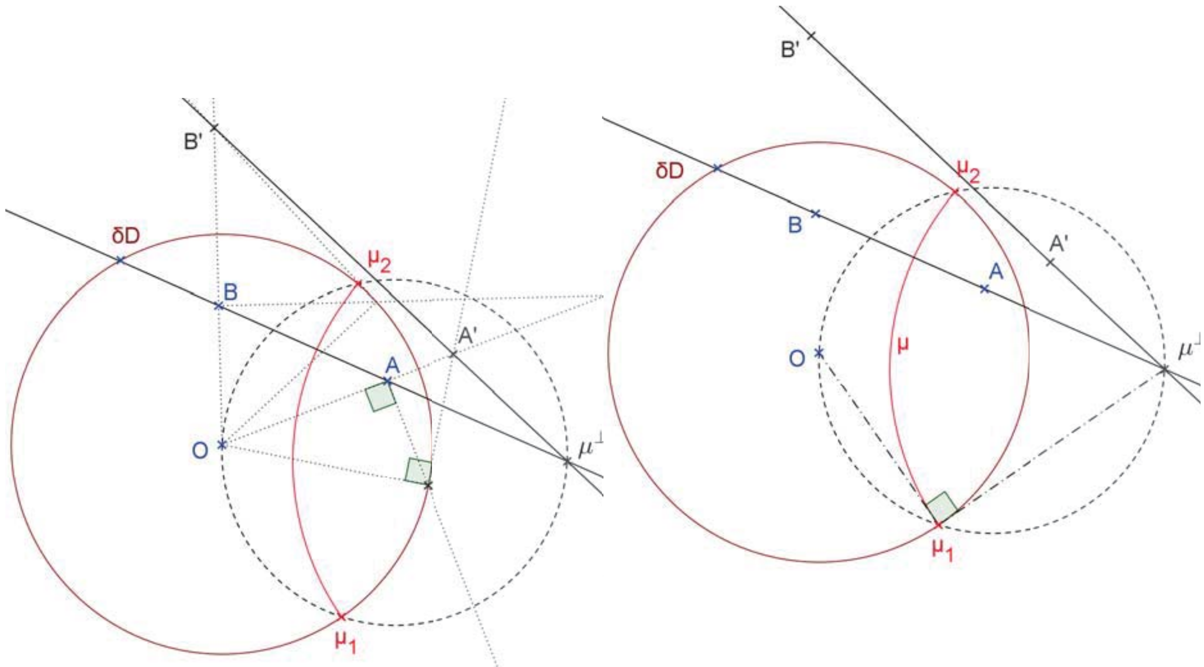

Figure 2.2: Construct the line bisector for two given points A and B.

Problem 2.2. *For two given points A and B, construct the perpendicular bisector μ.*

Hint: Let A', B' be the inverse points of A, B by ∂D. Both pairs A, B and A', B' are inverse points by circle μ.

Construction 2.2. *The Euclidean lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{A'B'}$ intersect in the point $\mu^\perp$. The perpendicular bisector in modelled by a circle around $\mu^\perp$ perpendicular to ∂D. One needs still to get the correct radius. The ideal endpoints μ_1, μ_2 of μ are the touching points of the tangents from $\mu^\perp$ to ∂D. These points are constructed via Thales' theorem. Indeed $\mu_{1,2}$ lie on a circle with diameter $O\mu^\perp$. Finally μ is the circle about $\mu^\perp$ through $\mu_{1,2}$.*

Problem 2.3. *Given is a hyperbolic line $\delta = \overleftrightarrow{EF}$, with ideal endpoints E and F, and a point P on the line δ. Use the given drawing, with $P, P^\perp, \delta^\perp, E, F$ already available to erect the perpendicular σ on the line δ at point P. Use the ideal endpoints S and T of σ to check the accuracy of your drawing. In this first variant, make use of the ideal endpoints.*

Construction 2.3. *The Euclidean lines $\overleftrightarrow{EF}$ and $P^\perp$ intersect at point $\sigma^\perp$.*

Finally, to get σ is easy, one simply draws a circular arc with center $\sigma^\perp$ through point P.

Remark 1. One need not even construct the tangent from $\sigma^\perp$ to ∂D. Since $\delta \perp \sigma$, it is known that the ideal endpoints S and T of σ and $\delta^\perp$ lie on a Euclidean line. This fact can be used to check the accuracy of the construction.

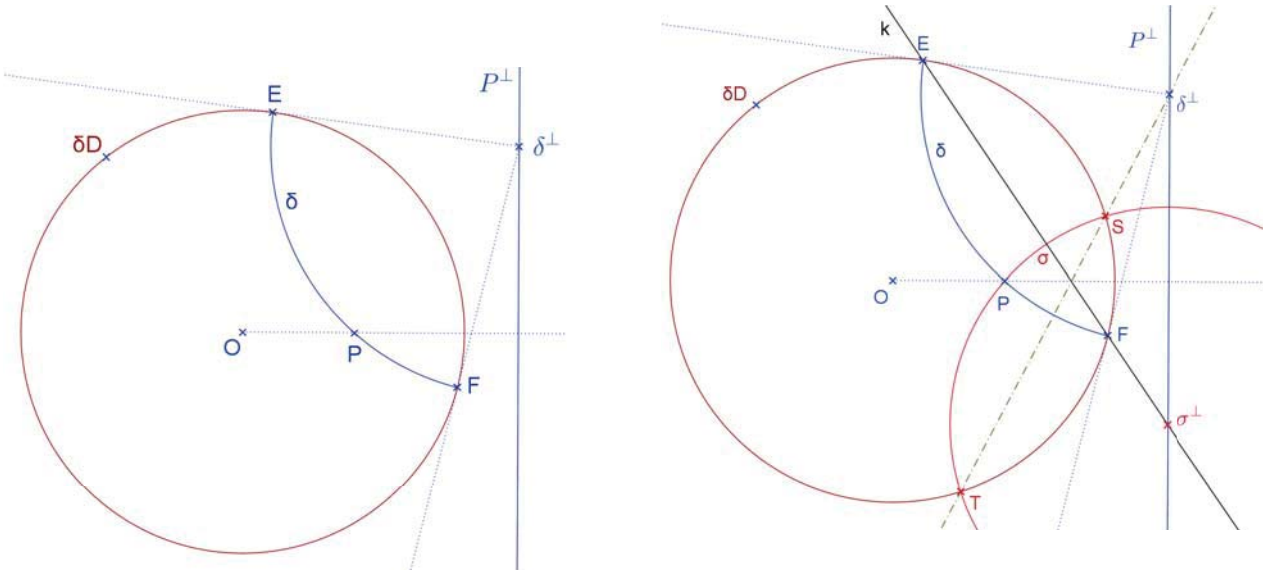

Figure 2.3: Erect a perpendicular on line δ at point P, using Construction 2.3.

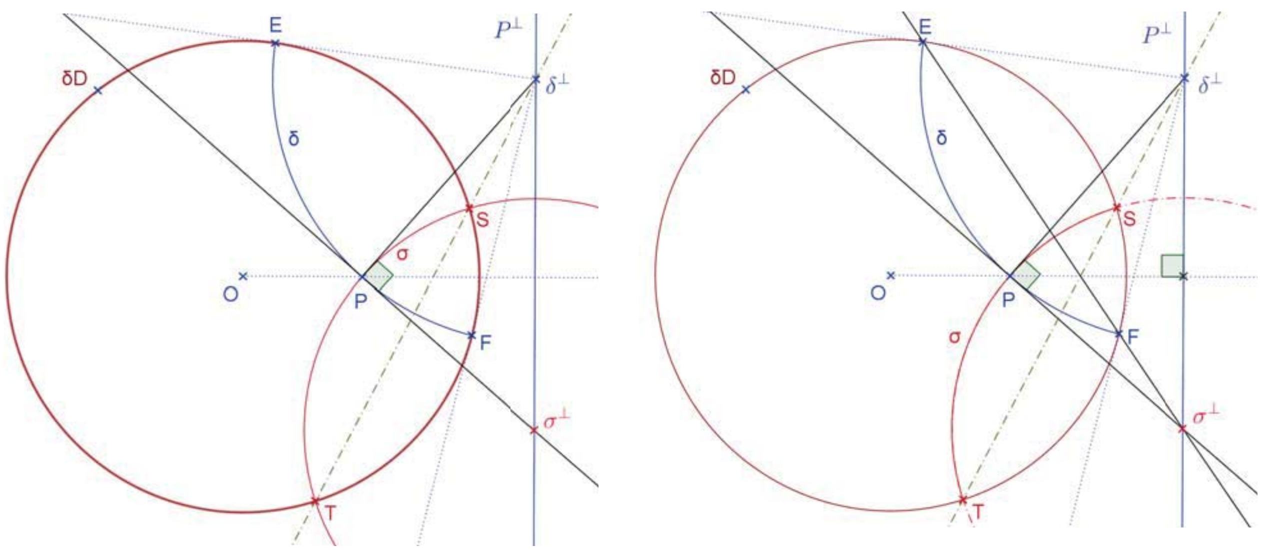

Figure 2.4: Erect a perpendicular on line δ at point P, using the right angle as explained in Remark 2. Finally, combining the two constructions yields better accuracy.

Remark 2. As an alternative construction, one can also erect the perpendicular on the radial ray $\overrightarrow{P\delta^\perp}$ at vertex P. Again, that perpendicular intersects line $p^\perp$ at the polar point $\sigma^\perp$. Using both constructions gives another possibility for better accuracy.

Problem 2.4. *Given is a hyperbolic line $\delta = \overleftrightarrow{EF}$, with ideal endpoints E and F, and a point P* not *on the line δ. Use the given drawing, with $P, P^\perp, \delta^\perp, E, F$ already available to erect the perpendicular σ on the line δ at point P. Use the ideal endpoints S and T of σ to check the accuracy of your drawing.*

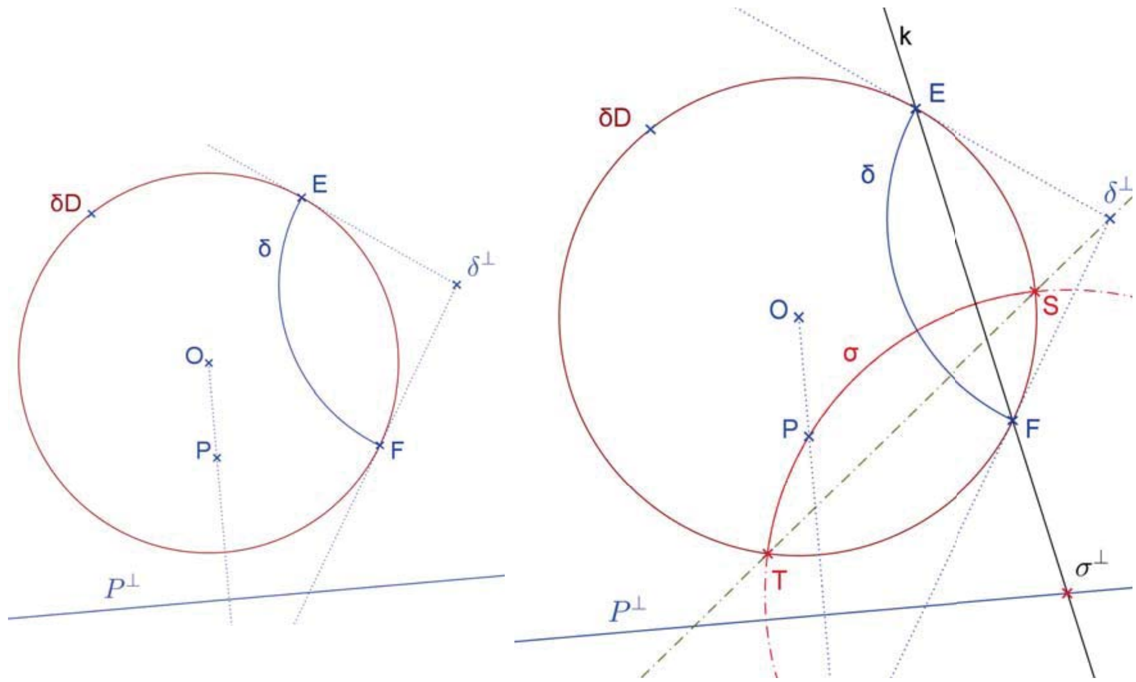

Figure 2.5: Drop a perpendicular onto line δ from the given point P.

Construction 2.4. *The Euclidean lines $\overleftrightarrow{EF}$ and $P^\perp$ intersect at point $\sigma^\perp$. Finally, to get σ is easy, one simply draws a circular arc with center $\sigma^\perp$ through point P.*

Problem 2.5. *Given is a hyperbolic line $\delta = \overleftrightarrow{EF}$, and a point P on the line δ, and an angle α. Use the given drawing, with $P, P^\perp, \delta^\perp$ already available. Construct two lines $\varepsilon_1, \varepsilon_2$ through point P that form the given angle α with the given line δ.*

Construction 2.5. *Transfer the given angle α onto both sides of the radial ray $\overrightarrow{P\delta^\perp}$ at vertex P. The two new sides produced by angles α intersect line $p^\perp$ in the polar points $\varepsilon_1^\perp, \varepsilon_2^\perp$. Now it is straightforward to get $\varepsilon_1^\perp, \varepsilon_2^\perp$, because one has already the point P on these lines. One simply draws two circular arcs with centers $\varepsilon_1^\perp$ and $\varepsilon_2^\perp$ through point P.*

Problem 2.6. *For two given intersecting lines α and β, construct an angle bisector δ. Explain how you can get the second bisector.*

Construction 2.6. *Let A_1, A_2 and B_1, B_2 be the ideal endpoints of lines α and β. One angular bisector δ has polar $\delta^\perp$ at the intersection of the Euclidean lines $\overleftrightarrow{A_1B_1}$ and $\overleftrightarrow{A_2B_2}$. The second angular bisector δ_2 has polar its $\delta_2^\perp$ at the intersection of the Euclidean lines $\overleftrightarrow{A_1B_2}$ and $\overleftrightarrow{A_2B_1}$. Finally one has to draw the hyperbolic lines δ and δ_2. They are modelled by circular arcs with centers $\delta^\perp$ and $\delta_2^\perp$ through the intersection point P of lines α and β.*

Alternative Construction. Let P be the intersection point of lines α and β. Let ε be the bisector of $\angle \alpha^\perp P \beta^\perp$, and let ε_2 be the outer bisector of that angle. (Indeed ε_2 is perpendicular to ε.) One angular bisector δ of the given lines α and β has its polar $\delta^\perp$

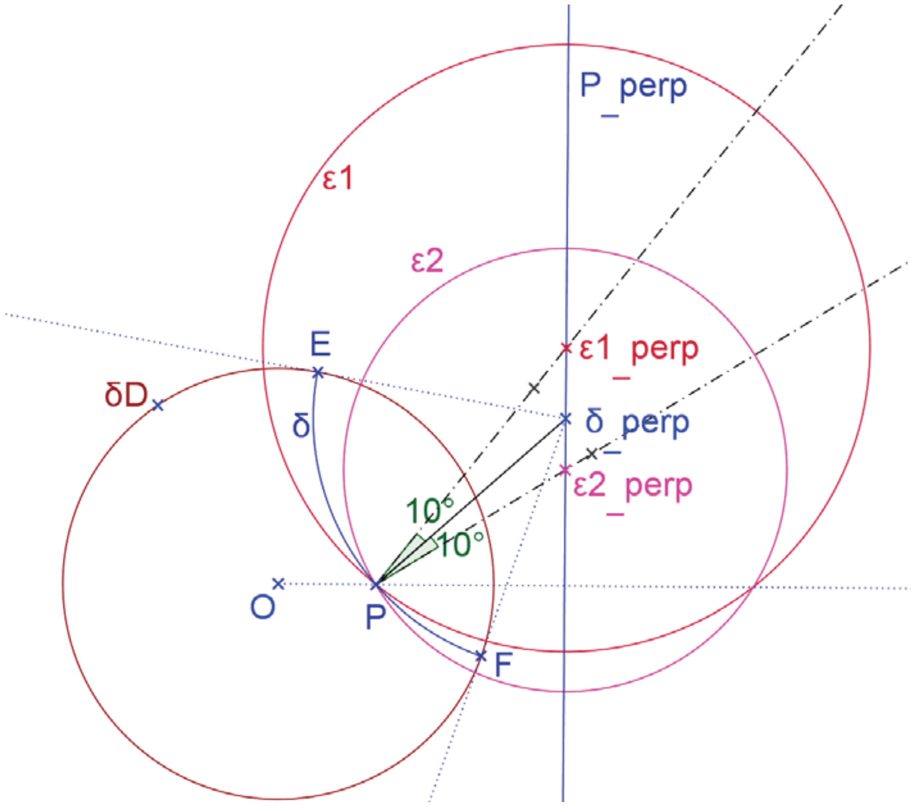

Figure 2.6: Transfer a given angle to a given ray. The example takes $\alpha = 10°$.

at the intersection of the Euclidean lines $\overleftrightarrow{\alpha^\perp\beta^\perp}$ and ε. The second angular bisector δ of lines α and β has its polar $\delta_2^\perp$ at the intersection of the Euclidean lines $\overleftrightarrow{\alpha^\perp\beta^\perp}$ and ε_2. $\square$

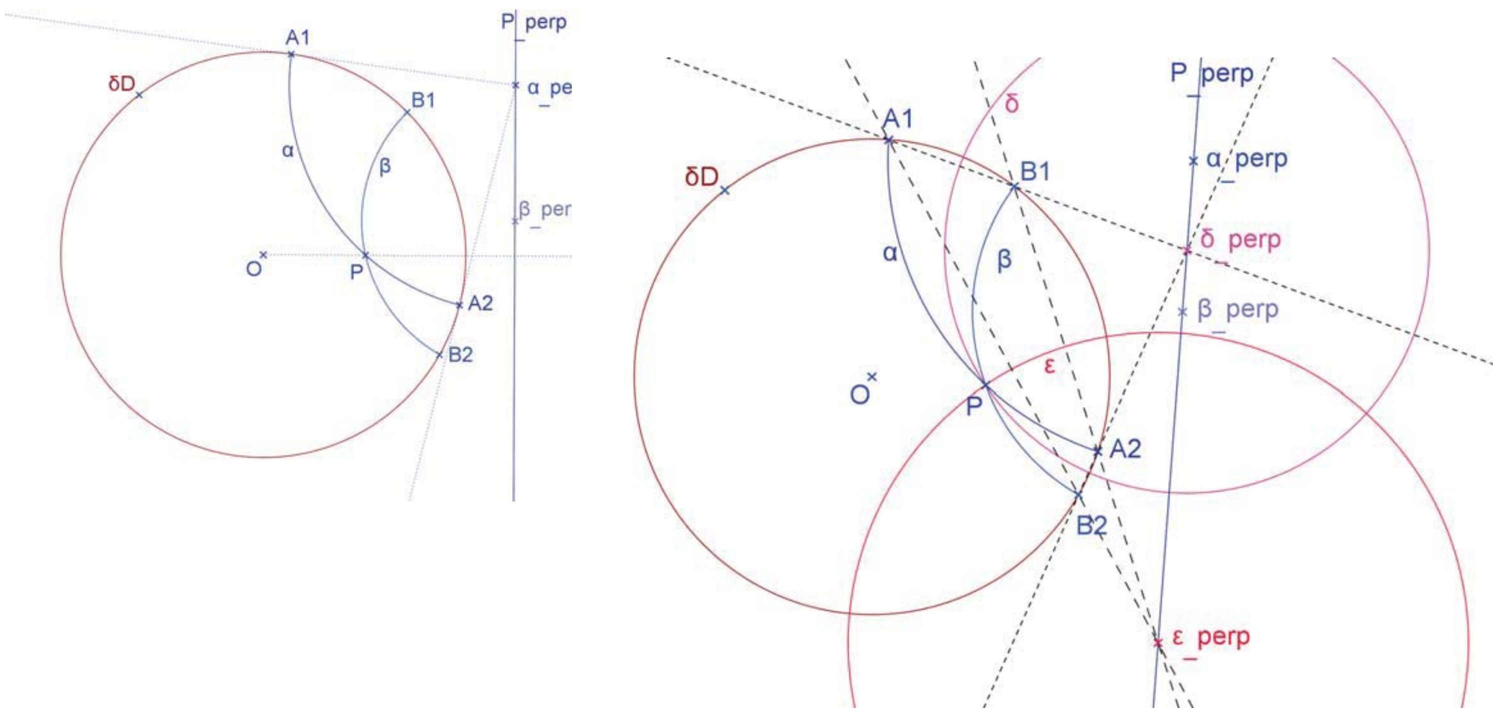

Figure 2.7: Construction of the angular bisectors.

2.2 Typically hyperbolic constructions

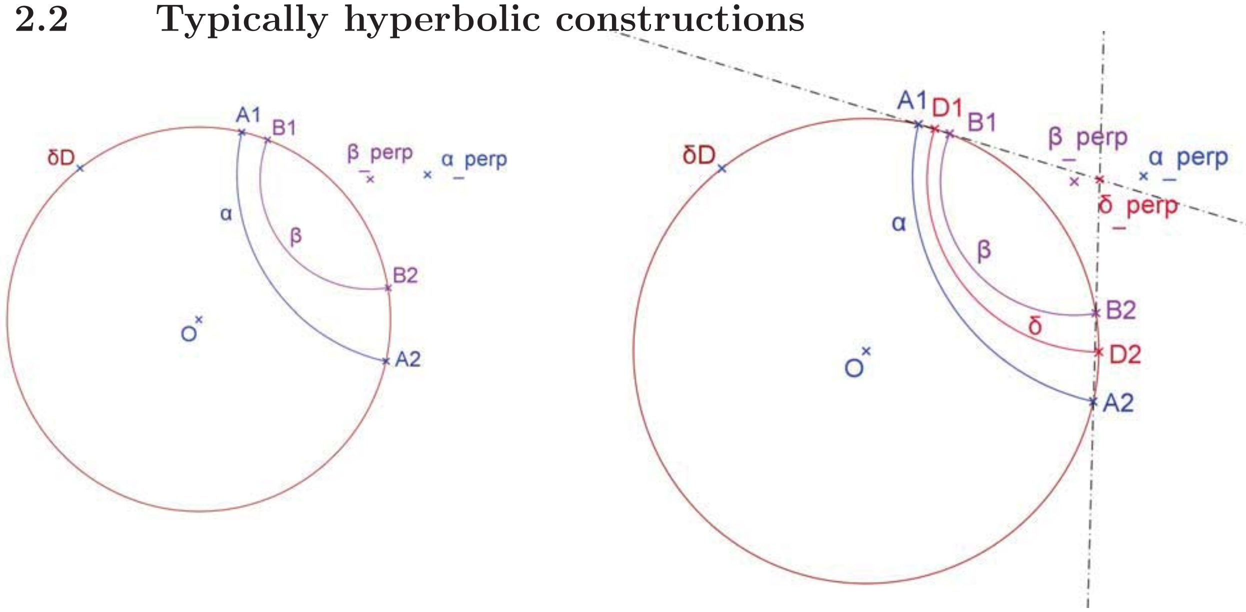

Figure 2.8: Construction of the middle line of two parallel lines.

Problem 2.7. *For two divergently parallel lines α and β, construct the middle line δ. The reflection by this line maps α to β.*

Construction 2.7. *Let $A_{1,2}$ and $B_{1,2}$ be the ideal endpoints of lines α and β. The numbering is such that, as one moves around the boundary circle δD, point A_1 lies adjacent to B_1, and point A_2 adjacent to B_2. The middle line δ has polar $\delta^\perp$ at the intersection of the Euclidean lines $\overleftrightarrow{A_1B_1}$ and $\overleftrightarrow{A_2B_2}$. (This intersection point lies* outside *the disk D, whereas the intersection of the Euclidean lines $\overleftrightarrow{A_1B_2}$ and $\overleftrightarrow{A_2B_1}$ lies* inside *D.)*

Problem 2.8. *For two divergently parallel lines α and β, construct the common perpendicular μ.*

Construction 2.8 (Variant 1)**.** *The common perpendicular μ has polar $\mu^\perp$ at the intersection of the Euclidean lines $\overleftrightarrow{A_1A_2}$ and $\overleftrightarrow{B_1B_2}$.*

Problem 2.9. *For two divergently parallel lines α and β, find a second way to construct the common perpendicular μ.*

Construction 2.9 (Variant 2)**.** *Draw the Euclidean line $c = \overleftrightarrow{\alpha^\perp\beta^\perp}$ connecting the polars of the two given lines. The M be the foot point of the perpendicular dropped from O onto this line c. The inverse point $M' = \mu^\perp$. It is constructed by putting the tangents to circle ∂D at the intersection points c with ∂D. Those two tangents and the perpendicular OM intersect all three in the point $\mu^\perp$.*

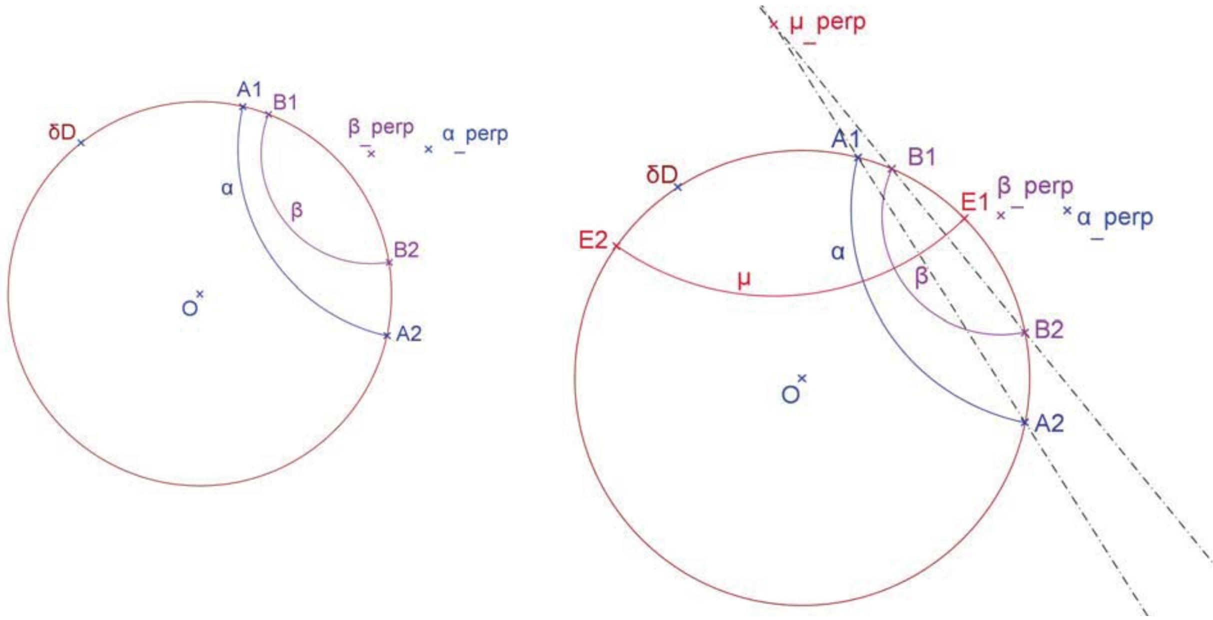

Figure 2.9: Construction of the common perpendicular of two parallel lines.

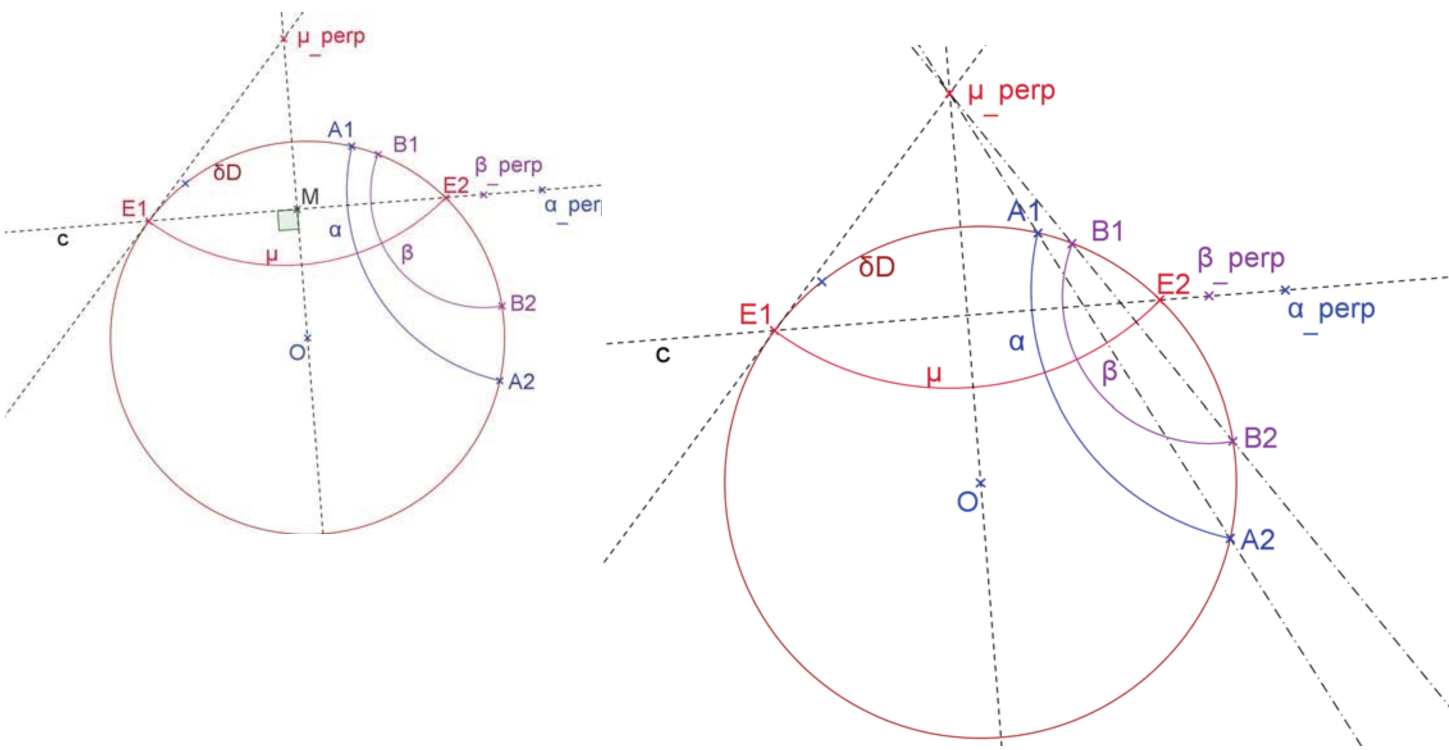

Figure 2.10: Variant 2 for the construction of the common perpendicular of two parallel lines— and a figure showing both variants.

Question. Describe and explain the simultaneous construction of the middle line and the common perpendicular of two parallel lines done in the figure on page 786.

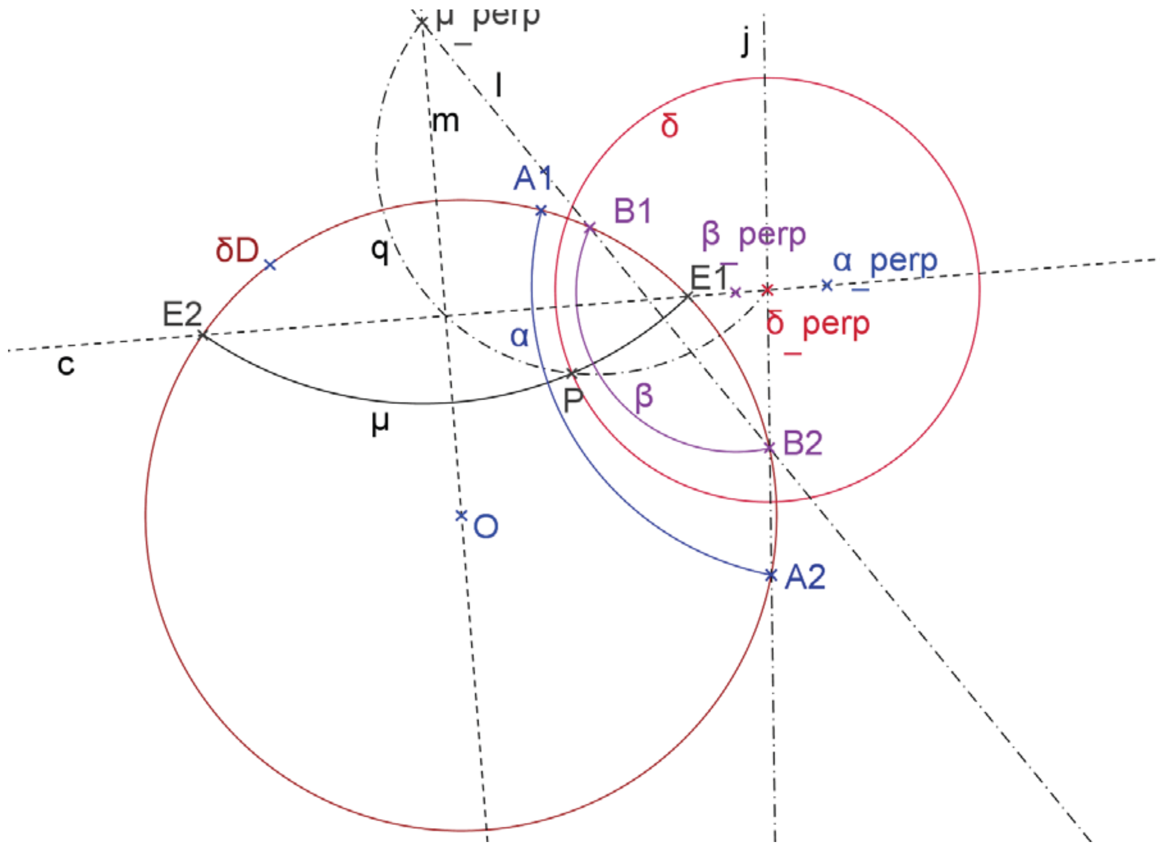

Figure 2.11: Given two parallel lines α and β, the middle line δ and the common perpendicular μ are constructed simultaneously.

2.3 Circle constructions

Problem 2.10. *For two points A and B, construct a circle ε with center A through point B.*

(a) *Assume the prescribed center $A \neq O$ and point B do not lie on the diameter OA. Moreover, assume the hyperbolic line $r = AB$ has already been constructed.*

(b) *Assume the prescribed center $A \neq O$ and point B lie on the diameter OA.*

Construction 2.10. *One needs to find the Euclidean quasi-center M of circle ε. It lays both on the segment OA, and on the tangent to the arc r at point B.*

(a) *In this case, these are two different lines which intersect in the quasi-center M.*

(b) *If center A, disk-center O and point B lie on a line—or even for a point B lying too close to the diameter OA—we construct at first a second point B" on the circle.*

To this end, we erect the hyperbolic perpendicular a to OA at point A, and reflect point B by line a. Since B and the reflection image B" have the same hyperbolic

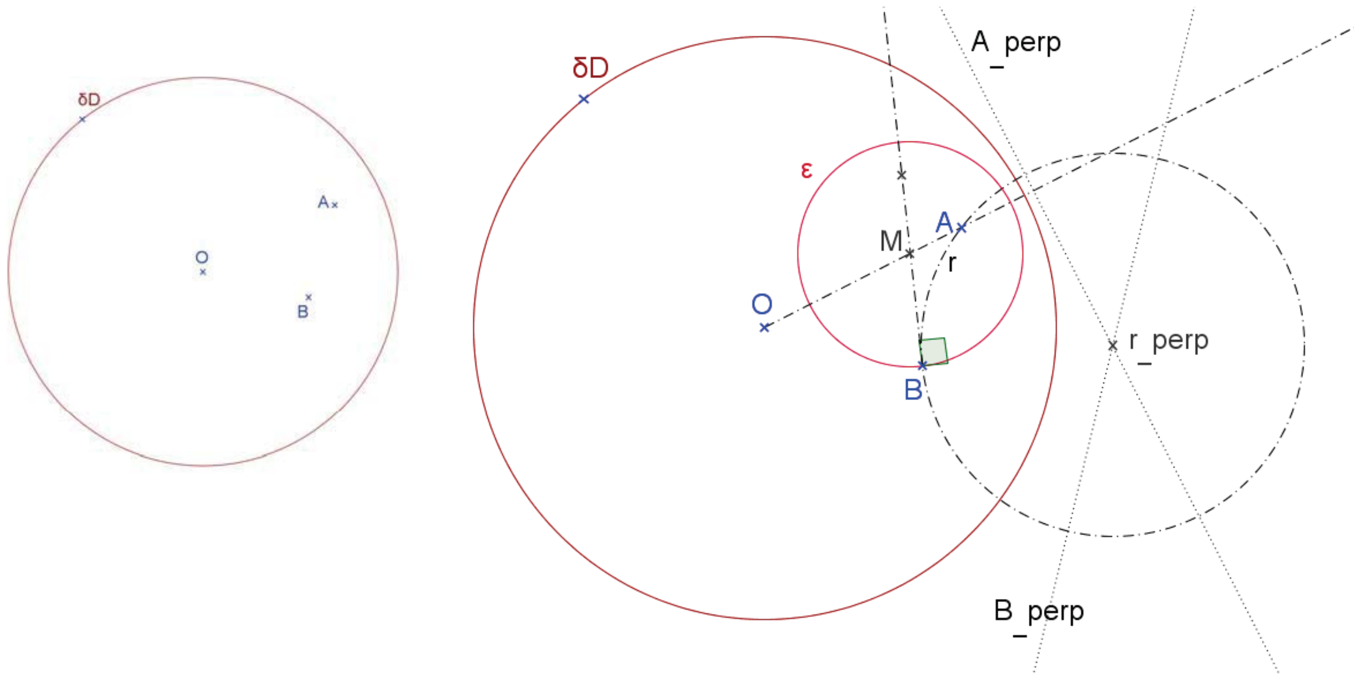

Figure 2.12: Construction of a circle with given center A through point B.

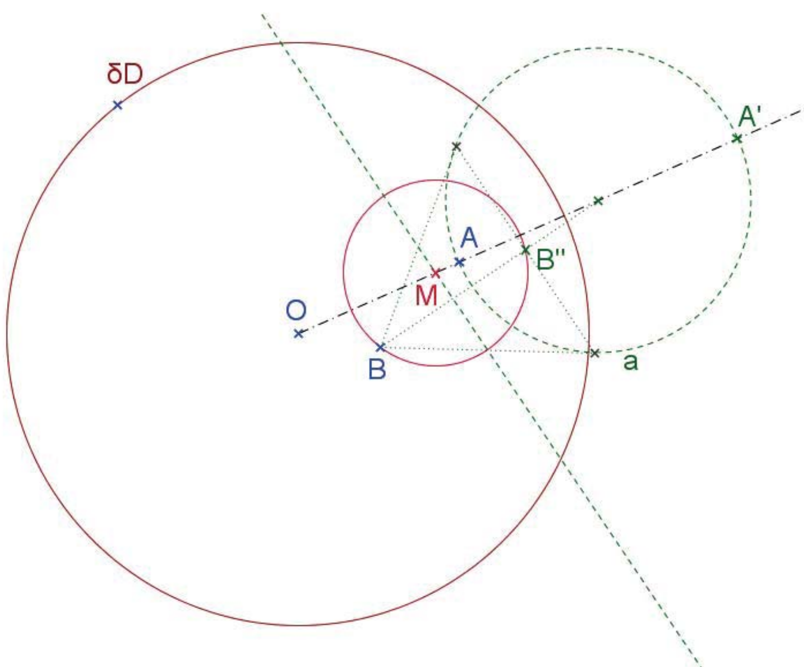

Figure 2.13: Construction of a circle with given center A through point B, which is on or close to line OA.

> *distance from A, both lie on the circle to be constructed. The circle is perpendicular to line OA, and hence its center M is the intersection of the perpendicular bisector of BB" with segment OA.*

Problem 2.11. *For two points A and B, construct a circle ε with center O and radius*

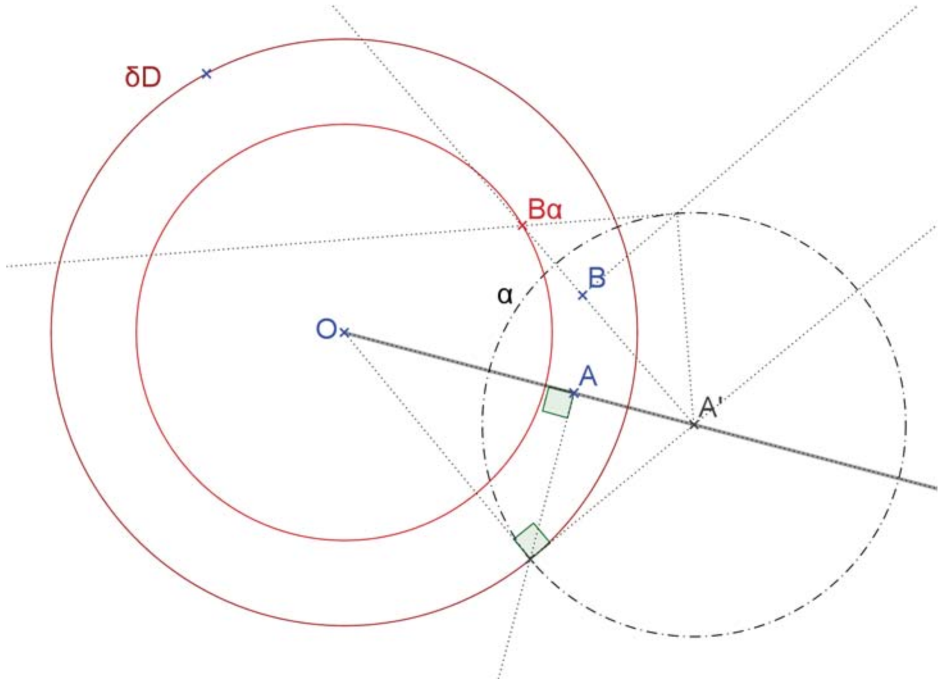

Figure 2.14: How to construct of a circle around O with given hyperbolic radius.

equal to the hyperbolic distance $s(A, B)$.

Construction 2.11. *Proceed as in the example "A useful reflection" to get the hyperbolic line* α*, the reflection by which maps point* A *to the center* O*. This reflection maps point* B *to image point* B_α*. One has to construct* B_α *via inversion by* α*. To this end, one draws the radial ray* $\overrightarrow{A'B}$*, and a perpendicular on it through point* B*, which intersect circle* α *in point* Q*. Then one erects the perpendicular on* $A'Q$ *at point* Q*. It intersect the ray* $\overrightarrow{A'B}$ *in the inversion point* B_α*. The circle around* O *through point* B_α *has the hyperbolic radius* $s(A, B) = s(A_\alpha, B_\alpha) = s(O, B_\alpha)$ *as required.*

2.4 Triangle constructions

Proposition 2.1 (Construction of a triangle from its three angles). *In hyperbolic geometry, its three angles determine a triangle up to congruence. Indeed, for any given angles α, β, γ with sum $\alpha+\beta+\gamma < 180°$, there exists a triangle, unique up to congruence.*

To achieve a construction in the Poincaré disk, it is convenient to chose the vertices first, and determine the disk D in a second step. Such a procedure does not introduce a real restriction, because, in an extra step, a linear delation can be used to map the entire figure such that the disk falls into any disk given in advance. Furthermore, we shall choose vertex C at the center of disk D.

Construction 2.12. *Let $\sigma = \alpha+\beta+\gamma$ be the angle sum and $\delta = 180° - \alpha - \beta - \gamma$ be the defect. One begins by drawing $\angle Ac^\perp B = \delta$ as given. Then draw a circle c with center $c^\perp$ of any radius. We can assume that A and B lie on that circle, and construct the tangents to c at A and B. Next, we lay off the angles α at vertex A, and β at vertex B, with intersecting tangent rays as one of their legs, and the second legs outside of circle c. Those second legs intersect at point $C = O$, and form an angle γ. Finally, one needs the boundary ∂D of the Poincaré disk. To this end, one constructs the tangents from point C to circle c. The touching points S, T of these tangents can be constructed via Thales' theorem. Indeed S and T lie on a circle with diameter $c^\perp C$. Finally the boundary ∂D is the circle about C through S and T. (Indeed, the points S, T are the ideal endpoints of the triangle side AB, too.)*

Problem 2.12. *Do the construction with given angles $\alpha = 40°, \beta = 50°, \gamma = 60°$.*

Proposition 2.2. *In hyperbolic geometry, its three angles determine a triangle up to congruence. Indeed, for any given angles α, β, γ with sum $\alpha+\beta+\gamma < 180°$, there exists a triangle, unique up to congruence. Too, there exist asymptotic triangles for which one, two, or even all three of its vertices are ideal endpoints. In the disk model, these ideal endpoints simply lie on ∂D. The angle at an ideal vertex is zero.*

Construction 2.13 (Construction of an asymptotic triangle from its two angles). *We assume that vertex C is ideal, and hence $\gamma = 0$. For simplicity, we take for vertex A the center of disk D. Hence side AC and the ray $\overrightarrow{AB}$ are hyperbolic as well as Euclidean lines. They form the given angle α at vertex A, and can be constructed immediately.*

Next we need to get $a^\perp$, the polar to the side $a = BC$. Since point C lies on the side a, the polar $C^\perp$ goes through the polar $a^\perp$. But the polar $C^\perp$ is simply the tangent to ∂D at the ideal point C. Thus we get already one coordinate for point $a^\perp$. We need to use the given angle β at vertex B to get a second coordinate.

Let $\sigma = \alpha+\beta+\gamma$ be the angle sum and $\delta = 180° - \alpha - \beta - \gamma$ be the defect. The Euclidean quadrilateral $\square ABa^\perp C$ has the angles

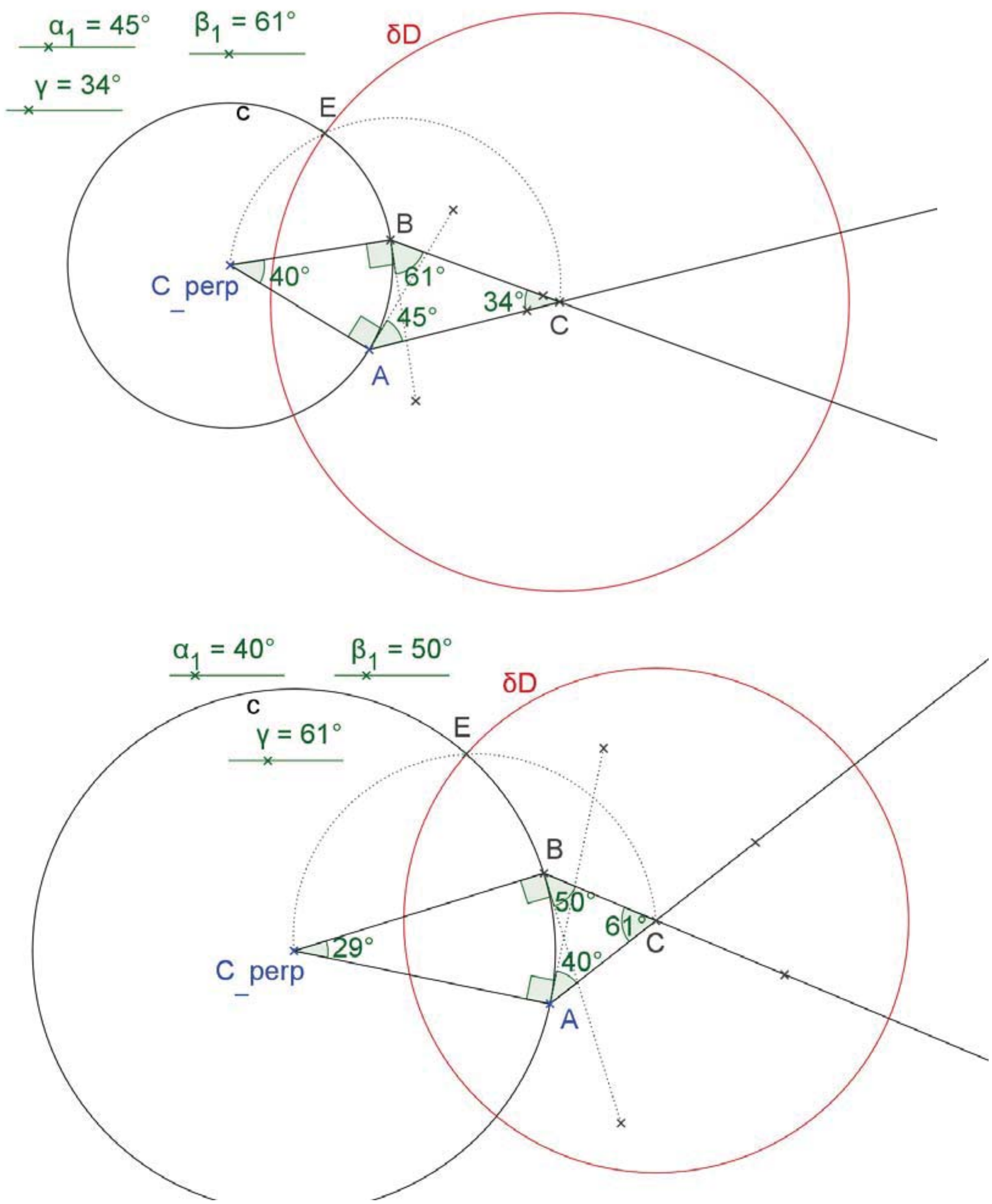

Figure 2.15: Two examples for the construction of a triangle from three given angles.

α at vertex A,

90° at vertex C,

$90^\circ + \beta$ at vertex B, and hence

δ at vertex $a^\perp$.

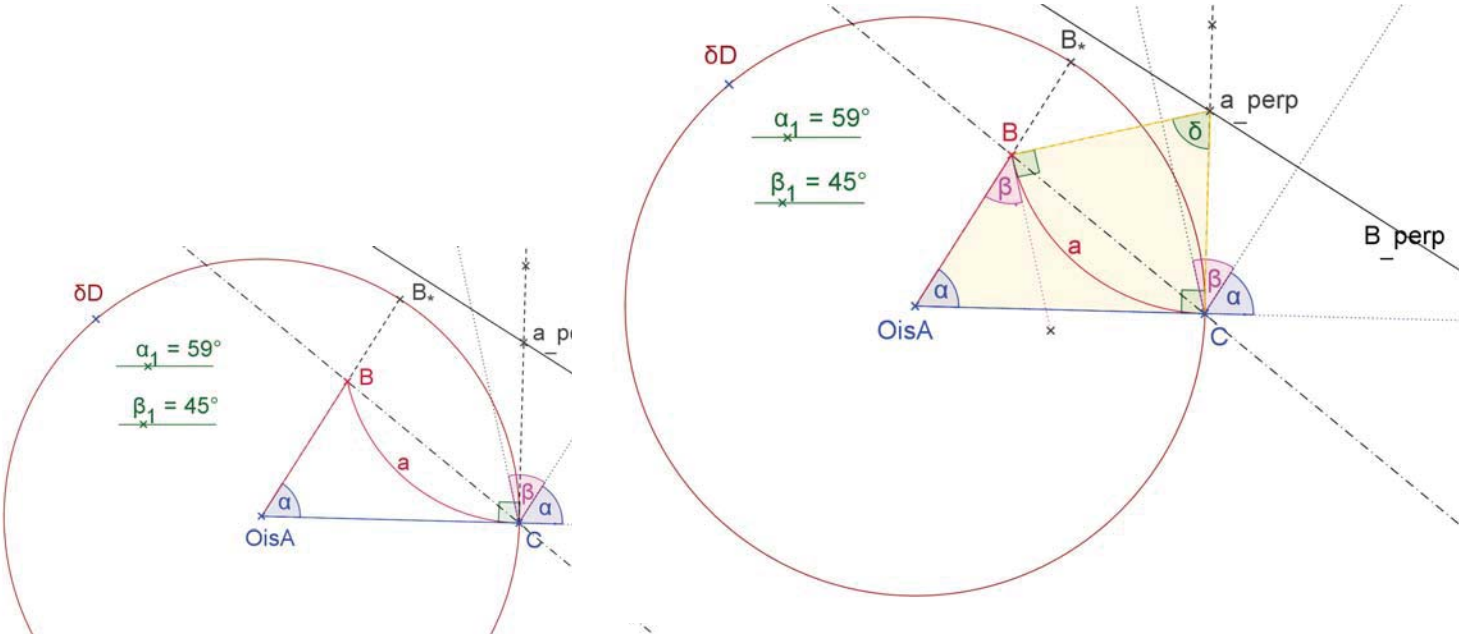

Figure 2.16: Construction of an asymptotic triangle with angles $\alpha = 60°, \beta = 45°, \gamma = 0°$.

Hence the isosceles $\triangle BCa^\perp$ has two base angles $\frac{\sigma}{2}$. Thus the ray $\overrightarrow{CB}$ form an angle $\frac{\sigma}{2}$ with the tangent $Ca^\perp$, and an angle $\frac{\delta}{2}$ with the radius CA.

Hence one transfers angle $\frac{\delta}{2}$ to vertex C with one side CA. Now point B lies on the other side of this angle.

Problem 2.13. *Do the construction with given angles $\alpha = 60°, \beta = 45°, \gamma = 0°$.*

Proposition 2.3 (SAA Construction of a triangle). *In hyperbolic geometry, a triangle is determined, up to congruence, by giving one side, one adjacent angle, and the angle opposite to that side. Indeed, for any given angles α, γ with sum $\alpha + \gamma < 180°$, and segment of length c, there exists such a triangle, unique up to congruence.*

I give two variants for a construction.

Explanation of construction variant 1. We choose to put the vertex B, where no angle is specified, at the center of the Poincaré disk. Let A be any point such that segment AB has the length c as required. The construction uses the inverse point A' and the polar $A^\perp$, too. Next we construct line b through point A, which produces angle α with $\overrightarrow{AB}$ as required. To this end, one transfers angle $90° - \alpha$ at vertex A with one side $\overrightarrow{AA'}$. The intersection of the other leg of this angle with $A^\perp$ yields point $b^\perp$. Now line b is simply a circular arc with center $b^\perp$ through point A.

The harder part is to get vertex C. Let C^* be the ideal endpoint of ray $\overrightarrow{AC}$ such that we get angle $\angle BAC^* = \alpha$ as required.

Transfer angle $90° + \gamma$ to vertex C^* with one side $\overrightarrow{C^*b^\perp}$. Let a^* be the other side of that angle. We have thus produced the given angle γ at vertex C^*, with one side a ray tangent to b, and ray a^* the other side. Let σ be the circle around $b^\perp$ through O and let O^* be the intersection point of that circle with the ray a^*.

The remaining part of the construction uses a Euclidean reflection by the bisector of angle $\angle Ob^\perp O^*$. Too, the reflection maps angle γ to vertex C, with one side a ray tangent to b, and the ray $a = CO$ as other side.

Answer.

Remark. Here is a slightly different way to get vertex C. One transfers angle γ to vertex A, with one leg being the tangent to side b, the other leg in the same half plane as $\overrightarrow{AB}$. (Alternatively, transfer angle $\gamma - \alpha$ with one leg $\overrightarrow{AB}$.) Let D be the intersection point of the second leg with σ, the circle around $b^\perp$ through point O. Finally, one has to draw a circle around B of Euclidean radius AD. It intersects the arc b in two points C_1 and C_2. One of the $\triangle ABC_i$ for $i = 1, 2$ has angle γ at vertex C_i. (The other one has the supplementary angle $180° - \gamma$, and has not to be taken into account.)

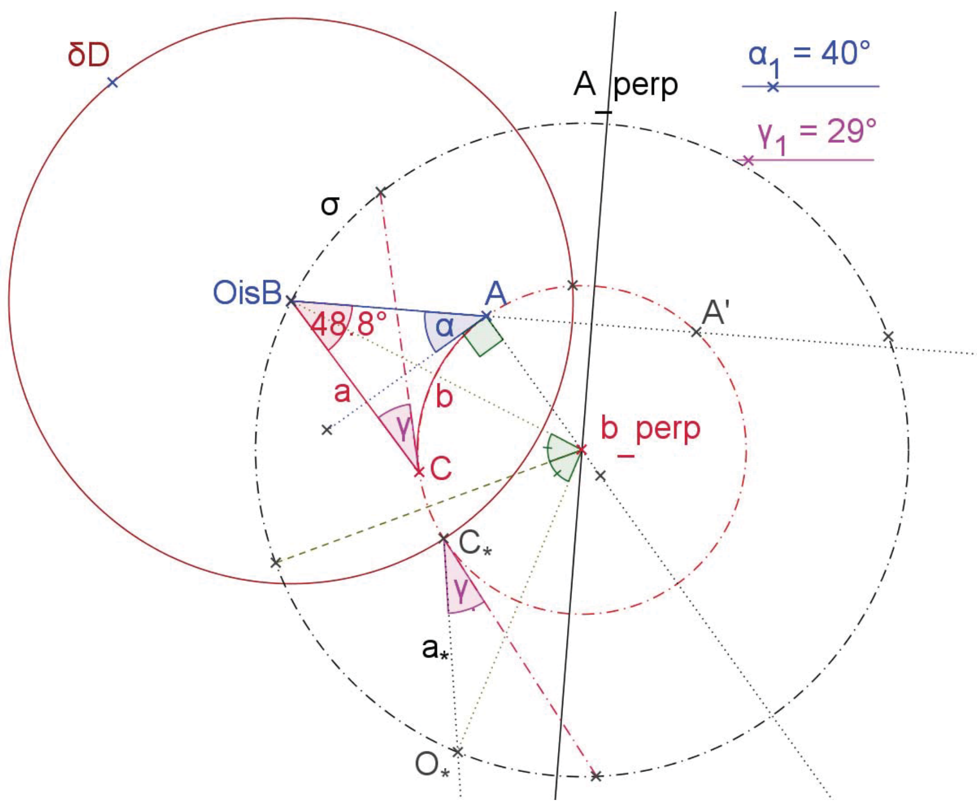

Figure 2.17: Example for the SAA construction of a triangle.

Explanation of construction variant 2. We choose to put the vertex A and the given angle α at the center $O = A$ of the Poincaré disk. Let B be any point such that segment AB has the length c as required. Let E be the ideal endpoint of ray $\overrightarrow{AC} = \overrightarrow{AE}$ such that we get angle $\angle BAE = \alpha$ as required, and F be the ideal end of the opposite ray.

The harder part is to get vertex B, to which end we use an equidistance line. By definition, an equidistance line is the set of all points with same distance from a given line. Let ε be the equidistance line through vertex B consisting of all points with same distance from line OE. The line ε is modelled by a circular arc though point B and the ideal ends E and F. as explained in the section on the Poincaré model.

We transfer angle γ to vertex $O = A$ with one side $\overrightarrow{OE}$. Let c^* be the other side of that angle, and B^* be its intersection with the equidistance line ε.

The remaining part of the construction uses a hyperbolic reflection by line r, which is constructed to map B to B^*, and the line FOE to itself. The polar $r^\perp$ is the intersection of the Euclidean lines BB^* and OE. Finally vertex C is the reflective image of O. This is the intersection of the common chord of δD and r with ray $\overrightarrow{OE}$.

Too, the reflection maps angle γ to vertex C, with the sides CO and CB.

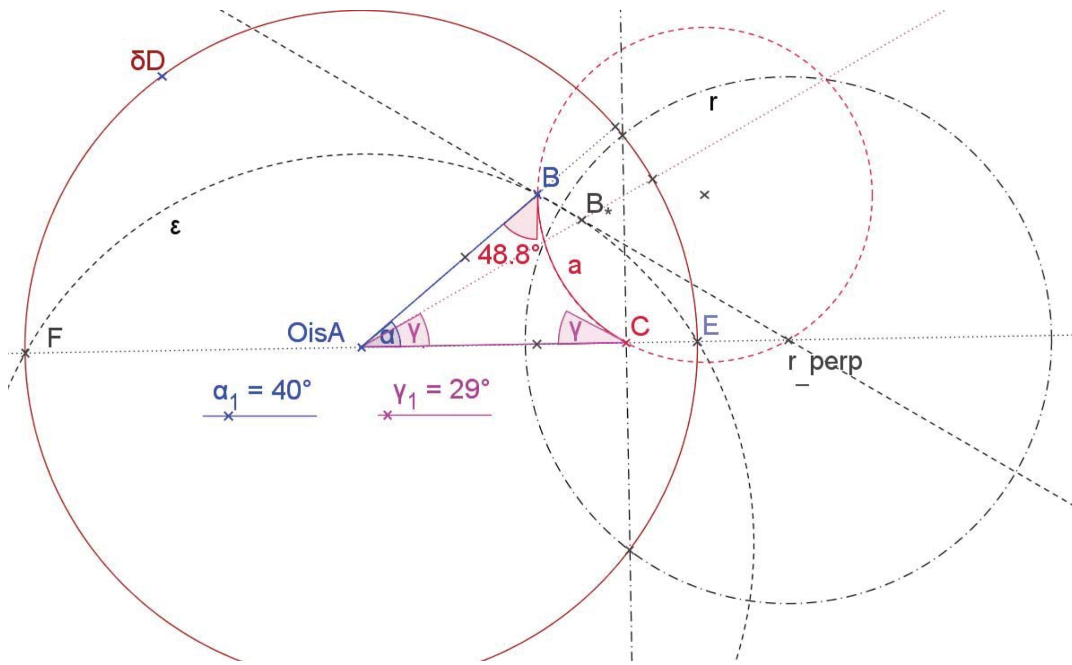

Figure 2.18: Variant 2 for the SAA construction. This variant uses an equidistant line and a hyperbolic reflection by r.

Problem 2.14. *We choose $\alpha = 40°$ and $\gamma = 29°$. For disk D and point A already given, do the construction as described above. Report the angle β from your construction.*

Remark. The actual value of β does depend to the choice of the hyperbolic distance $s(A, B)$.

2.5 The altitudes and the orthocenter

Problem 2.15. *Read the proof and of the proposition about the altitudes. Provide a drawing with all relevant entities named consistently.*

Proposition 2.4. *In hyperbolic geometry, if two altitudes of a triangle intersect, then all three altitudes intersect in one point.*

Proof. We use the Poincaré disk model of hyperbolic geometry. In the given $\triangle ABC$, let H be the intersection point of the two altitudes dropped from vertices A and C. By means of a hyperbolic reflection, we can put $H = O$ into the center of the Poincaré disk. As usual, let a, b, c denote the sides of the triangle opposite to its vertices A, B, C. We use the polar triangle with vertices $a^\perp, b^\perp, c^\perp$ polar to sides a, b, c. This is a triangle in the Euclidean sense. For an acute triangle, it has the Poincaré disk in its interior. (”swallowing that disk”)

The altitude h_A of $\triangle ABC$ dropped from A passes through the center of the Poincaré disk. Hence it is a Euclidean straight line. Since h_A is perpendicular to the opposite side a, it passes through the polar point $a^\perp$. Furthermore, line $h_A = \overleftrightarrow{OA}$ is perpendicular to $A^\perp = \overleftrightarrow{b^\perp c^\perp}$. Similarly, altitude h_C passes through $c^\perp$ and is perpendicular to $C^\perp$.

Hence lines h_A and h_C are the altitudes of the polar triangle $\triangle\ a^\perp, b^\perp, c^\perp$, too. We can use the fact, known from Euclidean geometry, that its altitudes intersect in one point, which is indeed H.

Hence line $b^\perp H$ is an altitude of the polar triangle and hence it is perpendicular to line $B^\perp$. Thus point B lies on line $b^\perp H$, which means that the three points $B, H, b^\perp$ lie on one straight line. This implies that side b of perpendicular to the altitude h_b, which is the altitude h_B of the original triangle, too, and passes through point H. $\square$

Remark. The three altitudes of an acute triangle always intersect. For an obtuse triangle, the altitudes may or may not intersect.

Problem 2.16. *Use the Poincaré model to construct a triangle with an orthocenter, but no circum-center— and another triangle without an orthocenter, but a circum-center.*

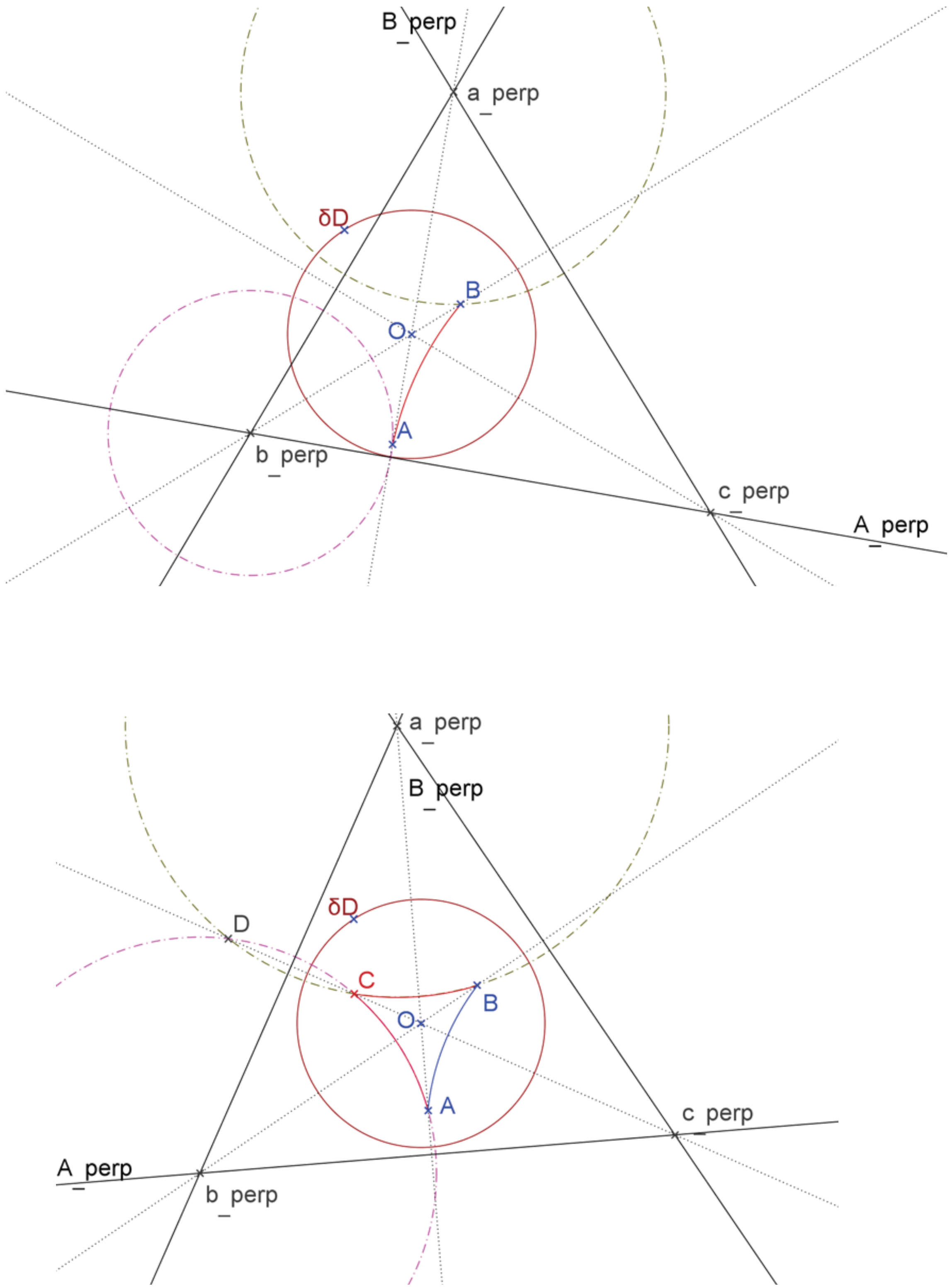

Figure 2.19: The three altitudes.

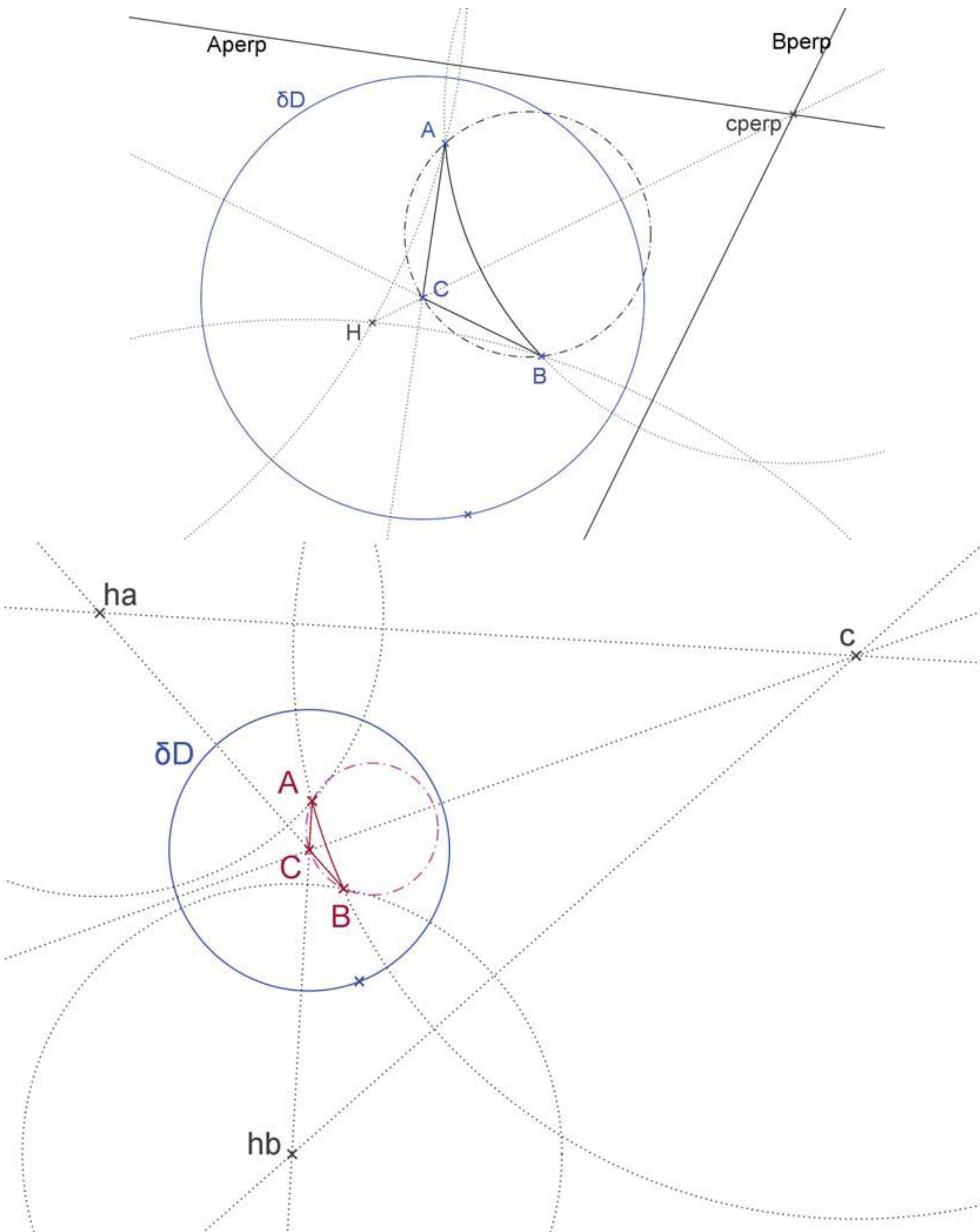

Figure 2.20: A triangle with an orthocenter, but no circum-center— another triangle without an orthocenter, but a circum-center.

3 Hyperbolic Geometry in Klein's Model

3.1 Setup of Klein's model

The second important model for hyperbolic geometry goes back to Felix Klein. The reader should recall the basic idea of a model in mathematics, as explained in the passage *General remark about models in mathematics*. Again, one uses the Euclidean plane as ambient underlying reality ("background ontology"). We put into the Euclidean plane the open unit disk

$$D = \{(x, y) : x^2 + y^2 < 1\}$$

with the boundary

$$\partial D = \{(x, y) : x^2 + y^2 = 1\}$$

The center of D is denoted by O.

Definition 3.1 (Basic elements of Klein's model). The points of D are the `"points"` for Klein's model. The points of ∂D are called `"ideal points"` or `"endpoints"`. The ideal points are not points of the hyperbolic plane. Once the hyperbolic distance is introduced, the points of ∂D turn out to be infinitely far away. Hence we call ∂D the `"circle of infinity"`. The `"lines"` for Klein's model are straight chords.

Poincaré's and Klein's model differ, because lines are represented differently, and—even more importantly—the hyperbolic isometries are given by different types of mappings. In Poincaré's model, the hyperbolic reflections are realized as inversions by circles. In Klein's model, the hyperbolic reflections are realized quite differently. Indeed, hyperbolic reflections are projective mappings, which leave the circle of infinity ∂D invariant.

The developing Klein's model based on projective geometry is postponed to the subsection about the projective nature of Klein's model. I shall now use a rather simple-minded different approach: there exists an isomorphism which is a translation from Poincaré's to Klein's model. Because we already know that Poincaré's model is a consistent model for hyperbolic geometry, the translation implies that Klein's model is a consistent model for hyperbolic geometry, too.

Proposition 3.1 (The mapping from Poincaré's to Klein's model). *The point P in Poincaré's model is mapped to a point K in Klein's model by requiring that the rays $\overrightarrow{OP} = \overrightarrow{OK}$ are identical and*

$$|OK| = \frac{2\,|OP|}{1 + |OP|^2} \tag{3.1}$$

The mapping (3.1) *keeps the ideal endpoints fixed, and it takes a circular arc $l \perp \partial D$ to the corresponding chord with the same ideal endpoints. Indeed, the mapping* (3.1) *is a* translation of Poincaré's to Klein's model *, since the points and lines of Poincaré's model, are mapped to points and lines of Klein's model, preserving incidence.*

Reason. As shown in the last proposition of the section on Poincaré's model, point K is the intersection of ray $\overrightarrow{OP}$ with the chord between the ideal endpoints of any arc $l \perp \partial D$ through point P. This chord k is a hyperbolic line in Klein's model. Clearly all points of arc l are mapped to points of chord k by the same construction, and hence are all given by mapping (3.1). $\square$

For the Poincaré disk model, it has been very useful to define *polar* elements outside the closed disk $\overline{D}$. We shall use polar elements for Klein's model, too. As a first step, we define and construct the *inverse point* P' of any given point P, in the way explained in the section about the Euclidean geometry of circles II.

Definition 3.2 (The polar elements for Klein's model). The *polar* $l^\perp$ of a line l is the intersection point of the tangents to ∂D at its ideal endpoints.

The *Klein polar* or *projective polar* $K^\perp$ of a point K is perpendicular to the ray $\overrightarrow{OK}$ at the inverse point K'.

A few clarifying remarks are in place: For both the Poincaré and the Klein model, the polar of a line is the intersection point of the tangents to the circle of infinity at the ideal ends.

But the mapping from the points to their polar elements are *different* for the two models. For clarification, I use the terms *Poincaré polar* and *projective polar.* [56] The difference occurs because the points are mapped from Poincaré's to Klein's via the isomorphism (3.1), but the polar elements are the same for both models, hence the mappings from points to their polar are different for the two models.

Proposition 3.2. *The Poincaré polar of a point P is the perpendicular bisector of the segment PP' between the given point and its inverse. The projective polar of a point K is the perpendicular to the ray $\overrightarrow{OK}$ at the inverse point K'.*

The following diagram for the mapping (3.1)*, the Poincaré polar and the projective polar is commutative:*

$$\begin{array}{ccc}
\textit{point } P \textit{ of Poincaré's model} & \xrightarrow{\textit{isomorphism (3.1)}} & \textit{point } K \textit{ of Klein's model} \\
{\scriptstyle \textit{perpendicular bisector of } PP'}\Big\downarrow & & \Big\downarrow{\scriptstyle \textit{perpendicular to } OK \textit{ at } K'} \\
\textit{Poincaré's polar } P^\perp & = & \textit{projective polar } K^{proj\perp}
\end{array}$$

Proof. By item (b) from the last proposition from the section on the Poincaré model, the inverse point K' is the midpoint of the segment PP' between P and its inverse point P'. Hence the Poincaré polar $P^\perp$ is identical to the Klein polar $K^{\text{proj}\perp}$. $\square$

[56] The simple term polar is common usage for the projective polar. It refers to the pole and polar relation studied in projective geometry. The term is also used in common software packages for geometry. On the other hand, Goodman-Strauss uses the term polar for the Poincaré polar, as I do in the sections on Poincaré's model.